(the "interesting part" is $N(H)$)

W_0 acts on H by $nhn^{-1} = h^n$ for $n \in N(H)$ $h \in H$

$\Rightarrow$ W_0 acts on $\mathfrak{h}$

$\Rightarrow$ W_0 acts on $\mathfrak{h}^*$

If M is a G-module, then $N(H)$ acts on M and

$$w : M_\mu \longrightarrow M_{w\mu}, \qquad w \in W_0 \qquad (1)$$

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{gen} \qquad (2)$$

(1), & (2) are tools for study repeesentatvans. So, if we understand (1) & (2) we know ~~e~~everything about M.

$SO_{10} = \{ g \in GL_{10} \mid \det g = 1,\ gg^t = 1 \}$

$so_{10} = \{ x\ \cancel{g} \in gl_{10} \mid tr\, x = 0,\ x + x^t = 0 \}$, since

$$1 = \det(e^{tx}) = \det \begin{pmatrix} e^{th_1} & & \\ & \ddots & \\ 0 & & e^{th_n} \end{pmatrix} = e^{t(h_1 + \cdots + h_n)} = e^{t\, tr(x)}$$

and $1 = e^{t(x)} (e^{tx^*})^t = e^{tx} e^{tx^t} = 1 + t(x + x^t) + O(t^2)$

1

So, so_{10} is the Lie algebra for SO_{10}.

So, $so_{10} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & & \\ -a_{12} & 0 & a_{23} & & \\ -a_{13} & -a_{23} & 0 & & \\ & & & \ddots & a_{9\,10} \\ & & & & 0 \end{pmatrix} \right\}$ $\Rightarrow$ $\dim(so_{10}) = 9 + 8 + \cdots + 1 = \frac{9 \cdot 10}{2} = 45$

Another choice is $SO_{10} = \{ g \in GL_{10} \mid \det g = 1,\ gJg^t = J \}$

$so_{10} = \{ x \in gl_{10} \mid tr\, x = 0 \quad xJ + Jx^t = 0 \}$

where,

$$J = \underbrace{\begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}}_{10} \quad \text{or} \quad J = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{smallmatrix} & 0 \end{array} \right)$$

It is more convenient to think this way. For example, SO_6.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1-3} & a_{1-2} & a_{1-1} \\ a_{21} & a_{22} & a_{23} & a_{2-3} & a_{2-2} & a_{2-1} \\ a_{31} & a_{32} & a_{33} & a_{3-3} & a_{3-2} & a_{3-1} \\ a_{-31} & a_{-32} & a_{-33} & a_{-3-3} & a_{-3-2} & a_{-3-1} \\ a_{-21} & a_{-22} & a_{-23} & a_{-2-3} & a_{-2-2} & a_{-2-1} \\ a_{-11} & a_{-12} & a_{-13} & a_{-1-3} & a_{-1-2} & a_{-1-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} a_{1-1} & a_{1-2} & a_{1-3} & a_{13} & a_{12} & a_{11} \\ a_{2-1} & a_{2-2} & a_{2-3} & a_{23} & a_{22} & a_{21} \\ a_{3-1} & a_{3-2} & \cdots & & & \\ \ddots & & & & \ddots & \end{pmatrix}.$$

So, $so_6 = \left\{ \begin{pmatrix} a_{11} & \underline{a_{12}} & \underline{a_{13}} & \underline{a_{1-3}} & \underline{a_{1-2}} & 0 \\ \boxed{a_{21}} & a_{22} & \underline{a_{23}} & \underline{a_{2-3}} & 0 & -a_{1-2} \\ \boxed{a_{31}} & \boxed{a_{32}} & a_{33} & 0 & -a_{2-3} & -a_{1-3} \\ a_{-31} & a_{-32} & 0 & -a_{33} & -a_{23} & -a_{13} \\ a_{-21} & 0 & -a_{32} & -a_{32} & -a_{22} & -a_{12} \\ 0 & -a_{-21} & -a_{-31} & -a_{31} & -a_{21} & -a_{11} \end{pmatrix} \right\}$

$$h = \left\{ \begin{pmatrix} a_{11} & & & & & \\ & a_{22} & & & 0 & \\ & & a_{33} & & & \\ & & & -a_{33} & & \\ & 0 & & & -a_{22} & \\ & & & & & -a_{11} \end{pmatrix} \right\}$$

For so_{10}, $h = \left\{ \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & 0 & \\ & & h_5 & & & \\ & & & -h_5 & & \\ & 0 & & & \ddots & \\ & & & & & -h_1 \end{pmatrix} \right\}$ (a Cartan subalgebra)

So, so_{10} has basis: $X_{\varepsilon_i - \varepsilon_j} = E_{ij} - E_{-j,-i}$ —

$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i}$ ‥

for $1 \le i < j \le 5$

$X_{-(\varepsilon_i - \varepsilon_j)} = E_{ji} - E_{-i,-j}$ $\square$

$X_{-(\varepsilon_i + \varepsilon_j)} = E_{-ij} - E_{-j,i}$ $\triangle$

(i.e. $X_{\varepsilon_1 - \varepsilon_2} = E_{12} - E_{-2-1}$

So, $\mathfrak{so}_{10} = \mathfrak{h} + \sum_{1\le i<j\le 5} a_{ij} X_{\varepsilon_i-\varepsilon_j} + a_{i-j} X_{(\varepsilon_i+\varepsilon_j)} + a_{ji} X_{(\varepsilon_i+\varepsilon_j)}$

$$+ \; a_{ji} X_{-(\varepsilon_i+\varepsilon_j)}$$

$(*)$ $$= \mathfrak{h} + \sum_{1\le i<j\le 5} \mathbb{C} X_{(\varepsilon_i-\varepsilon_j)} + \mathbb{C} X_{(\varepsilon_i+\varepsilon_j)} + \mathbb{C} X_{-(\varepsilon_i+\varepsilon_j)} + \mathbb{C} X_{-(\varepsilon_i-\varepsilon_j)}$$

then $[h, X_{\varepsilon_i-\varepsilon_j}] = [h, E_{ij} - E_{-j,-i}] = h(E_{ij} - E_{-j-i}) - (E_{ij} - E_{(-j-i)})h$.
$$= (h_i - h_j) E_{ij} - (-h_j + h_i) E_{-j-i}$$
$$= (h_i - h_j) X_{\varepsilon_i-\varepsilon_j} = (\varepsilon_i - \varepsilon_j)(h) X_{\varepsilon_i-\varepsilon_j}$$

where $\varepsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$ for $i = 1, 2, \ldots, 5$
$h \longrightarrow h_i$

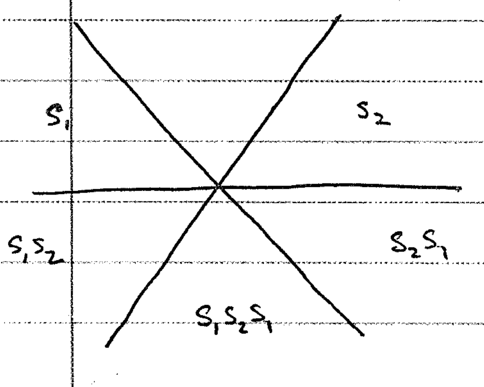
So, $\mathfrak{h}$ acts on $X_{\varepsilon_i-\varepsilon_j}$ the representation / eigenvalue $\varepsilon_i - \varepsilon_j$
So $(*)$ is a decomposition of $\mathfrak{so}_{10}$ is eigenspaces for the action of $\mathfrak{h}$ (adjoint action).
The root system
$R = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le 5\}$
$R^+ = \{(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le 5\}$ ← corresponds to upper triangle part of $\mathfrak{so}_{10}$.

So $\mathfrak{so}_{10} = \mathfrak{h} \oplus \sum_{\alpha\in R^+} \mathbb{C} X_\alpha + \mathbb{C} X_{-\alpha}$.

The functions $\varepsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$ forms a basis of $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$

$$\begin{pmatrix} h_1 & & & & & 0 \\ & \ddots & & & & \\ & & h_5 & & & \\ & & & -h_5 & & \\ & & & & \ddots & \\ 0 & & & & & -h_1 \end{pmatrix} \mapsto h_i$$

So, $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_5\}$. Inside $\mathfrak{h}^*_{\mathbb{R}}$ are hyperplanes



$(W_0, \mathfrak{h}^*_{\mathbb{Z}})$
$\mathbb{R}^2 = \mathfrak{h}^*_{\mathbb{R}}$

$\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee} = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \varepsilon_i^\vee - \varepsilon_j^\vee \rangle = 0\}$

corresponds to SL_3

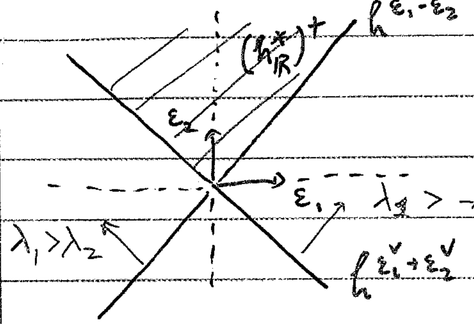
$\mathfrak{h}^{\varepsilon_i^\vee+\varepsilon_j^\vee} = \{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \varepsilon_i^\vee + \varepsilon_j^\vee \rangle = 0 \}$ $\quad 1 \le i < j \le 5$

where $\langle \varepsilon_i^\vee, \varepsilon_j^\vee \rangle = \delta_{ij}$.

For so_4, $\mathfrak{h}^{\varepsilon_1^\vee-\varepsilon_2^\vee} = \{ \lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 \mid \langle \lambda, \varepsilon_1^\vee - \varepsilon_2^\vee \rangle = 0 \}$

$= \{ \lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 \mid \lambda_2 = \lambda_1 \}$

So, we get the following picture.



$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^2$

Back to so_{10},

$\mathfrak{h}^*_{\mathbb{R}} = \{ \lambda_1\varepsilon_1 + \dots + \lambda_5\varepsilon_5 \mid \lambda_1, \dots, \lambda_5 \in \mathbb{R} \}$

with ~~eigen~~ hyperplanes:

$\mathfrak{h}^{\varepsilon_i^\vee-\varepsilon_j^\vee}$ and $\mathfrak{h}^{\varepsilon_i^\vee+\varepsilon_j^\vee}$, $1 \le i < j \le 5$

Therefore, the fundamental chamber is

$$(\mathfrak{h}^*_{\mathbb{R}})^+ = \{ \lambda_1\varepsilon + \dots + \lambda_5\varepsilon_5 \mid \langle \lambda, \varepsilon_i^\vee - \varepsilon_j^\vee \rangle \geq 0 \text{ and } \langle \lambda, \varepsilon_i^\vee + \varepsilon_j^\vee \rangle \geq 0 \text{ for } 1 \le i \le j \le 5 \}.$$

$$= \{ \lambda_1\varepsilon_1 + \dots + \lambda_5\varepsilon_5 \mid \lambda_1 > \lambda_2 > \dots > |\lambda_5| \}$$

W_0 is generated by reflections on $\mathfrak{h}^{\varepsilon_i^\vee-\varepsilon_j^\vee}$ and $\mathfrak{h}^{\varepsilon_i^\vee+\varepsilon_j^\vee}$ for $1 \le i < j \le 5$.

Let $s_{\varepsilon_i^\vee-\varepsilon_j^\vee}$ be the reflection on $\mathfrak{h}^{\varepsilon_i^\vee-\varepsilon_j^\vee}$ that swithes ε_i and ε_j

" $s_{\varepsilon_i^\vee+\varepsilon_j^\vee}$ be " " " $\mathfrak{h}^{\varepsilon_i^\vee+\varepsilon_j^\vee}$ " " ε_j and $-\varepsilon_j$

So, $W_0 \simeq$ 5×5 matrices with (a) exactly one non-zero entry in each row and each column. (symmetric group)

(b) the nonzero entrées are ± 1. (second cond.)

(c) $\prod(\text{non zero entries}) = 1$.

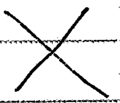
i.e.
$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in W_0$$

Recall that the Dynkin diagram is the dual graph of walls of the chamber. i.e. vertices are the walls of chamber $(\mathfrak{h}^*_{\mathbb{R}})^+$

edges are
- $\overset{i}{\circ} \quad \overset{j}{\circ}$ if $\mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$ is $\pi/2$ $(\underline{\mathfrak{h}})$
- $\overset{i}{\circ}\!-\!\!\!-\overset{j}{\circ}$ " " " is $\pi/3$
- $\overset{i}{\circ}=\overset{j}{\circ}$ " " " is $\pi/4$
- $\overset{i}{\circ}\equiv\overset{j}{\circ}$ " " " is $\pi/6$

i.e. $\circ\!-\!\!\!-\circ$ for SL_3.

For SO_4



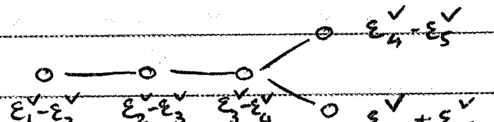
$\overset{1:1}{\longleftrightarrow} \quad \circ \quad \circ$

For SO_{10} $(\mathfrak{h}^*_{\mathbb{R}})^+ = \{ \lambda_1\varepsilon_1 + \dots + \lambda_5\varepsilon_5 \mid \lambda_1 > \lambda_2 > \dots > \lambda_5$ and $\lambda_4 > -\lambda_5 \}$

has walls

$\mathfrak{h}^{\varepsilon_1^\vee - \varepsilon_2^\vee}, \mathfrak{h}^{\varepsilon_2^\vee - \varepsilon_3^\vee}, \mathfrak{h}^{\varepsilon_3^\vee - \varepsilon_4^\vee}, \mathfrak{h}^{\varepsilon_4^\vee - \varepsilon_5^\vee}$ and $\mathfrak{h}^{\varepsilon_4^\vee + \varepsilon_5^\vee}$.

This corresponds to the Dynkin diagram



use $\langle u, v\rangle = |u||v| \cos(u \measuredangle v)$

The adjoint representation of $\mathfrak{g} = so_{10}$.

i.e.

$\mathfrak{g}$ acts on $\mathfrak{g}$ by $ad_y : \mathfrak{g} \to \mathfrak{g}$ for $y \in \mathfrak{g}$

$x \longmapsto [y, x]$

If M is a $\mathfrak{g}$-module, the <u>character</u> of M is

$$\mathrm{char}(M) = \sum_{\mu \in \mathfrak{h}^*} \dim(M_\mu)\, e^\mu \quad \text{, where.}$$

$M_\mu = \{ m \in M \mid$ for each $h \in \mathfrak{h}$, $hm = \mu(h)m \}$

For so_{10}

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_4} \oplus \dots \oplus \mathfrak{g}_{-(\varepsilon_4 + \varepsilon_5)}$$

where $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}_\alpha = \mathbb{C} X_\alpha$ for $\alpha \in R = \{ \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le$

So, $\mathrm{char}(\mathfrak{g}) = 5e^0 + e^{\varepsilon_1+\varepsilon_2} + e^{\varepsilon_1+\varepsilon_3} + \dots + e^{-\varepsilon_4-\varepsilon_5}$.

Let $x_i = e^{\varepsilon_i}$. Then $\mathrm{char}(\mathfrak{g})$ is

$$s_{\varepsilon_1+\varepsilon_2} = 5 + x_1x_2^{-1} + x_1x_3^{-1} + x_1x_4^{-1} + \dots + x_4^{-1}x_5^{-1}$$

Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha = \frac{1}{2}\Big((\varepsilon_1-\varepsilon_2 + \varepsilon_1-\varepsilon_3 + \varepsilon_1-\varepsilon_4 + \varepsilon_1-\varepsilon_5 + \varepsilon_1+\varepsilon_5 + \varepsilon_1+\varepsilon_4$
$\varepsilon_1+\varepsilon_3 + \varepsilon_1+\varepsilon_2) + (\varepsilon_2-\varepsilon_3 + \varepsilon_2-\varepsilon_4 + \dots) + \dots\Big)$

$$= \tfrac{1}{2}(8\varepsilon_1 + 6\varepsilon_2 + 4\varepsilon_3 + 2\varepsilon_4)$$
$$= 4\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$$

So $e^\rho = e^{4\varepsilon_1+3\varepsilon_2+2\varepsilon_3+\varepsilon_4} = x_1^4x_2^3x_3^2x_4$

$$a_\rho = \sum_{w\in W_0} \det(w)\, w e^\rho = x_1^4x_2^3x_3^2x_4 - x_1^{-4}x_2^3x_3^2x_4 + \dots$$

$$a_{\varepsilon_1+\varepsilon_2+\rho} = \sum_{w\in W_0} \det(w)\, w e^{\varepsilon_1+\varepsilon_2+\rho} = \sum_{w\in W_0} \det(w)\, w(x_1x_2x_1^4x_2^3x_3^2x_4)$$

Weyl character formula says: $s_{\varepsilon_1+\varepsilon_2} = \dfrac{a_{\varepsilon_1+\varepsilon_2+\rho}}{a_\rho}$ (Amazing!)

Back to $\mathfrak{g}$.

$\mathfrak{g} = so_{10} = \mathfrak{h} \oplus \mathfrak{g}_{\varepsilon_1+\varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_1-\varepsilon_3} \oplus \dots$ where.
$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = \mathbb{C}X_{\varepsilon_i-\varepsilon_j}$ and $\mathfrak{g}_{\varepsilon_i+\varepsilon_j} = \mathbb{C}X_{\varepsilon_i+\varepsilon_j}$

~~Crystal~~ Crystals. are set of paths in $\mathfrak{h}_{\mathbb{R}}^* \simeq \mathbb{R}^5$
with an action of the root operators.
$$\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{f}_5$$

The crystal $B(\varepsilon_1+\varepsilon_2)$ corresponding the adjoint representation of so_{10} has
40 straight line paths $p_{\pm(\varepsilon_i\pm\varepsilon_j)}$ (p_λ is straight line from 0 to λ)

5 paths $\frac{1}{2} P_{-\varepsilon_1+\varepsilon_2} \otimes \frac{1}{2} P_{\varepsilon_1-\varepsilon_2}$, ..., $\frac{1}{2} P_{-\varepsilon_4-\varepsilon_5} \otimes \frac{1}{2} P_{\varepsilon_4+\varepsilon_5}$

(these end at 0)

Note: The points $\pm\varepsilon_i \pm \varepsilon_j$ are some of the vertices of a 5-dim cube.

Standard model in particle physics.

We want to decompose $B(\varepsilon_1+\varepsilon_2)$ (adjoint of so_{10}) for the subgroups

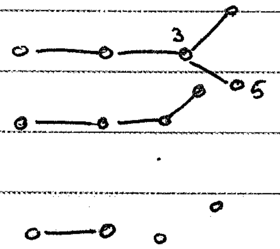
SO_{10} =

$\cup|$

SU_5 = — drop the 5th node.

$\cup|$

$SU_3 \times U_1 \times SU_2$ = drop the 3rd node.



For us, this means ignore $\tilde{f}_3$, $\tilde{f}_5$, $\tilde{e}_3$, $\tilde{e}_5$.

When you this $B(\varepsilon_1+\varepsilon_2)$ decomposes into ~ 10 connected components. Because of "symmetry breaking" only those components symmetric about 0 correspond to particles.

There are 3 such components of size 3, 1, and 8

bosons W^+, Z, W^- — photon — gluons of QCD