Representation Theory 01.04.2009

Tensor product of matrices

If $A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ \vdots & & \vdots \\ b_{s1} & \cdots & b_{ss} \end{pmatrix}$

action on $M = \mathrm{span}\{m_1, \ldots, m_r\}$ and $N = \mathrm{span}\{n_1, \ldots, n_s\}$ then $M \otimes N$ has basis

$$m_1 \otimes n_1, \ldots, m_1 \otimes n_s, m_2 \otimes n_1, \ldots, m_2 \otimes n_s, \ldots, m_r \otimes n_1, \ldots, m_r \otimes n_s$$

and if

$$(A \otimes B)(m_i \otimes n_j) = (Am_i \otimes Bn_j) \quad \text{then}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ \vdots & & & \vdots \\ a_{r1}B & & \cdots & a_{rr}B \end{pmatrix}$$

The quantum group $U_q sl_2$ is generated by $E, F, K^{\pm 1}$ with relations

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

and coproduct $\Delta : U \to U \otimes U$ given by

$$\Delta(E) = E \otimes K + 1 \otimes E,$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \qquad \Delta(K) = K \otimes K.$$

For $U_q\mathfrak{sl}_2$:

$L(\square) = \mathrm{span}\{v_1, v_{-1}\}$ with

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad K \longmapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

and

$L(\underbrace{\square\cdots\square}_{k \text{ boxes}}) = \mathrm{span}\{v_k, v_{k-2}, \ldots, v_{-(k-2)}, v_{-k}\}$ with

$$E \mapsto \begin{pmatrix} 0 & [k] & & & \\ & 0 & [k-1] & & \\ & & 0 & \ddots & \\ & & & 0 & [1] \\ & & & & 0 \end{pmatrix} \qquad F \mapsto \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & [2] & 0 & & \\ & & \ddots & \ddots & \\ & & & [k] & 0 \end{pmatrix}$$

$$K \longmapsto \begin{pmatrix} q^k & & & & \\ & q^{k-2} & & & \\ & & \ddots & & \\ & & & q^{-(k-2)} & \\ & & & & q^{-k} \end{pmatrix}$$

where $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$.

The <u>quantum</u> spin <u>chain</u> is

$L(\square)^{\otimes k} = \mathrm{span}\{v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_k} \mid \epsilon_j \in \{\pm 1\}\}$

and $U_q\mathfrak{sl}_2$ acts on $L(\square)^{\otimes k}$

For example

$$E(v_1 \otimes v_1) = 0$$

$$F(v_1 \otimes v_1) = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}$$

$$F^2(v_1 \otimes v_1) = 0 + q\, v_{-1} \otimes v_{-1} + q^{-1} v_{-1} \otimes v_{-1} + 0 = [2]\, v_{-1} \otimes v_{-1}$$

$$F^3(v_1 \otimes v_1) = 0$$

and $E(v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}) = F(v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}) = 0.$

Define an action of $TL_2 = \mathrm{span}\{1\!\!1, \overset{\cup}{\cap}\}$ on $L(\square)^{\otimes 2}$ by

$$\overset{\cup}{\cap}(v_1 \otimes v_1) = 0, \qquad \overset{\cup}{\cap}(v_{-1} \otimes v_{-1}) = 0$$

$$\overset{\cup}{\cap}(v_1 \otimes v_{-1}) = q\, v_1 \otimes v_{-1} - v_{-1} \otimes v_1$$

$$\overset{\cup}{\cap}(v_{-1} \otimes v_1) = q^{-1} v_{-1} \otimes v_1 - v_1 \otimes v_{-1}.$$

As matrices we have

$$\rho^{\otimes 2}(\overset{\cup}{\cap}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho^{\otimes 2}(\overset{\cup}{\cap})^2 = [2]\,\rho^{\otimes 2}(\overset{\cup}{\cap})$$

so that this is an action of TL_2 on $L(\square)^{\otimes 2}$.

The TL_2 action <u>commutes</u> with the $U_q\mathfrak{sl}_2$ action. i.e.

$$\rho^{\otimes 2}(TL_2) \subseteq \mathrm{End}_{U}(L(\square)^{\otimes 2}).$$

The Temperley-Lieb algebra TL_k is generated by

$$e_j = |\,|\,|\,|\overset{i\ \ i+1}{\overset{\cup}{\cap}}|\,|\,|\,|, \qquad 1 \le j \le k-1$$

Define an action of TL_k on $L(\square)^{\otimes k}$ by

$$e_j(v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_k}) = v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_{j-1}}\otimes \overset{\cup}{\cap}(v_{\varepsilon_j}\otimes v_{\varepsilon_{j+1}})\otimes v_{\varepsilon_{j+2}}\otimes\cdots\otimes v_{\varepsilon_k}$$

Claim: (a) This defines a TL_k action on $V^{\otimes k}$

(b) This TL_k action commutes with the $U_q\mathfrak{sl}_2$ action on $V^{\otimes k}$.

Let A be an algebra and let M be a semisimple A-module

$$M = \bigoplus_{\lambda\in\hat{A}} (A^\lambda)^{\oplus m_\lambda}$$

Let

$$Z = \mathrm{End}_A(M).$$

Then

$$Z = \bigoplus_{\lambda\in\hat{M}} M_{m_\lambda}(\mathbb{C}) \quad\text{and}\quad M \cong \bigoplus_{\lambda\in\hat{M}} A^\lambda\otimes Z^\lambda$$

as (A,Z) bimodules where

$$\hat{M} = \{\lambda\in\hat{A} \mid m_\lambda \neq 0\}.$$

Proof

$$Z = \mathrm{Hom}_A(M, M)$$

$$= \mathrm{Hom}_A\Big(\bigoplus_{\lambda \in \hat{M}} \bigoplus_{i=1}^{m_\lambda} A_i^\lambda, \bigoplus_{\mu \in \hat{M}} \bigoplus_{j=1}^{m_\mu} A_j^\mu\Big)$$

$$= \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i,j=1}^{m_\lambda} \mathrm{Hom}_A(A_i^\lambda, A_j^\mu)$$

by Schur's Lemma. Hence

$Z = \mathrm{span}\{e_{ij}^\lambda \mid \lambda \in \hat{M}, 1 \le i,j \le m_\lambda\}$ where

$$e_{ij}^\lambda : A_i^\lambda \longrightarrow A_j^\lambda.$$

Normalize the e_{ij} so that $(e_{ii}^\lambda)^2 = e_{ii}^\lambda$ and e_{ij}^λ and e_{ji}^λ so that $e_{ij}^\lambda e_{ji}^\lambda = e_{ii}^\lambda$. $\square$