Representation Theory 05.05.2009.

Spaces

A Lie group is a group G that is a manifold such that the maps

$$\begin{array}{l} G \times G \to G \\ (g_1, g_2) \mapsto g_1 g_2 \end{array} \quad \text{and} \quad \begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array}$$

are morphisms of manifolds.

An algebraic group is a group G that is also a variety such that the maps

$$\begin{array}{l} G \times G \to G \\ (g_1, g_2) \mapsto g_1 g_2 \end{array} \quad \text{and} \quad \begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array}$$

are morphisms of varieties

A topological group is a group G that is also a topological space such that the maps

$$\begin{array}{l} G \times G \to G \\ (g_1, g_2) \mapsto g_1 g_2 \end{array} \quad \text{and} \quad \begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array}$$

are morphisms of topological spaces.

A group scheme is a group G that is also a scheme such that $\begin{array}{l} G \times G \to G \\ (g_1, g_2) \mapsto g_1 g_2 \end{array}$ and $\begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array}$

are morphisms of schemes

A <u>complex</u> <u>Lie group</u> is a group G that is also a complex manifold such that

$$\begin{array}{ccc} G\times G \to G & & G \to G \\ (g_1, g_2) \mapsto g_1 g_2 & \text{and} & g \mapsto g^{-1} \end{array}$$

are morphisms of complex manifolds

<u>Remarks</u> (a) morphisms of manifolds = smooth functions

(b) morphisms of varieties = regular functions

(c) morphisms of topological spaces = continuous functions

(d) manifolds are spaces locally isomorphic to $\mathbb{R}^n$

(e) varieties are spaceces locally isomorphic to an affine variety.

(f) schemes are spaces locally isomorphic to an affine scheme.

(g) schemes are varieties over $\mathbb{Z}$.

(h) complex manifolds are not manifolds.

There are equivalences of categories

{connected reductive complex algebraic groups}	G
$\updownarrow$	$\updownarrow$
{connected compact Lie groups}	K
$\updownarrow$	$\updownarrow$
{complex semisimple Lie algebras}	$\mathfrak{g}$
$\updownarrow$	$\updownarrow$
{$\mathbb{Z}$-reflection groups}	$(W_0, \mathfrak{h}_{\mathbb{Z}}^*)$
$\updownarrow$	
{Dynkin diagrams}	

Examples GL_n, SL_n, PGL_n

$$GL_n(\mathbb{C}) = \{ g \in M_n(\mathbb{C}) \mid g \text{ is invertible} \}$$

Let V be a vector space over $\mathbb{F}$.

$$GL(V) = \{ g \in End(V) \mid g \text{ is invertible} \}$$

The group homomorphism

$$det : GL_n(\mathbb{F}) \longrightarrow \mathbb{F}^{\times}$$

is a 1-dimensional representation (character) of $GL_n(\mathbb{F})$.

$$SL_n(\mathbb{F}) = ker(det) = \{ g \in GL_n(\mathbb{F}) \mid det(g) = 1 \}$$

The centers of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are

$$Z(GL_n(\mathbb{C})) = \{ c \cdot Id \mid c \in \mathbb{C}^{*} \} = \mathbb{C}^{\times} \cdot Id.$$

$$Z(SL_n(\mathbb{C})) = \{ n^{th} \text{ roots of } 1 \} = \mu_n$$

Define

$$PGL_n(\mathbb{F}) = \frac{GL_n(\mathbb{F})}{Z(GL_n(\mathbb{F}))}$$

$GL_n(\mathbb{C})$ is a complex reductive algebraic group

$SL_n(\mathbb{C})$ is a complex semisimple algebraic group

$PGL_n(\mathbb{C})$ is a complex semisimple algebraic group.

In spite of

$$SL_n(\mathbb{C}) \subseteq GL_n(\mathbb{C}) \quad \text{and} \quad GL_n(\mathbb{C}) = SL_n(\mathbb{C}) \cdot \mathbb{C}^\times$$

and

$$1 \to SL_n(\mathbb{C}) \to GL_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^* \to 1$$

being exact

$$PGL_n(\mathbb{C}) \not\cong SL_n(\mathbb{C}).$$

<u>Examples</u> U_n, O_n, Sp_{2n}

The <u>unitary group</u>

$$U_n = \{ g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

The <u>orthogonal group</u>

$$O_n = \{ g \in GL_n(\mathbb{C}) \mid gg^t = 1 \}$$

The <u>symplectic group</u>

$$Sp_{2n}(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid gJg^t = J \}$$

where

$$\bar{g} = (\bar{g}_{ij}) \quad \text{and} \quad J = \left(\begin{array}{c|c} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & \end{array}\right) \quad \text{or} \quad J = \left(\begin{array}{c|c} & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & \end{array}\right)$$