Representation Theory 07.04.2009

sl_2 crystals

Start with

$B(a) = \{\rightarrow, \leftarrow\}$ with an action of $\tilde{e}$ and $\tilde{f}$

$\tilde{e}(\rightarrow) = 0, \quad \tilde{f}(\rightarrow) = \leftarrow$

$\tilde{e}(\leftarrow) = \rightarrow, \quad \tilde{f}(\leftarrow) = 0$

$$\begin{array}{c} \uparrow \tilde{e} \\ \rightarrow \\ \tilde{f}\downarrow\uparrow\tilde{e} \\ \leftarrow \\ \tilde{f}\downarrow \end{array}$$

Tensor products are by concatentation:

$B(a)\otimes B(a) = \{\rightarrow\rightarrow, \rightleftarrows, \leftrightarrows, \leftarrow\leftarrow\}$ and

$B(a)\otimes B(a)\otimes B(a) = \{\rightarrow\rightarrow\rightarrow, \rightarrow\rightleftarrows, \leftarrow\rightleftarrows, \rightleftarrows\!\!\rightarrow, \leftrightarrows\!\rightarrow, \leftrightarrows\!\leftarrow, \leftarrow\leftrightarrows, \leftarrow\leftarrow\leftarrow\}$

and the action of $\tilde{f}$ on $p\otimes q$ is

$$\tilde{f}(p\otimes q) = \begin{cases} \tilde{f}p\otimes q, & \text{if the (last occurence) of the most negative point is in } p, \\ p\otimes \tilde{f}q, & \text{if the (last occurence of the) most negative point of } p\otimes q \text{ is in } q. \end{cases}$$

and the action of $\tilde{e}$ ~~is~~ given by

$$\tilde{e}b = \begin{cases} b', & \text{if } \tilde{f}b' = b \\ 0, & \text{otherwise} \end{cases}$$

Decomposing $B^{\otimes k}$ where $B = B(\square)$

$$B(\square)\otimes B(\square) = \{\rightarrow\rightarrow,\ \leftrightarrows,\ \rightleftarrows,\ \leftarrow\leftarrow\}$$

$$\begin{array}{c} \rightarrow\rightarrow \\ \tilde f \downarrow \\ \rightleftarrows \\ \tilde f \downarrow \\ \leftarrow\leftarrow \end{array} \qquad \text{and} \qquad \begin{array}{c} \uparrow \tilde e \\ \leftrightarrows \\ \tilde f \downarrow \end{array} \qquad \text{so that}$$

$$B(\square)\otimes B(\square) \simeq B(\square\square) \sqcup B(\phi), \text{ where}$$

$B(\square\square) = \{\rightarrow\rightarrow,\ \rightleftarrows,\ \leftarrow\leftarrow\}$ and $B(\phi) = \{\leftrightarrows\}$.

Then

$$B(\square)\otimes B(\square)\otimes B(\square) = (B(\square\square)\sqcup B(\phi))\otimes B(\square)$$

$$= (B(\square\square)\otimes B(\square)) \sqcup (B(\phi)\otimes B(\square)) \quad \text{and}$$

$$B(\square\square)\otimes B(\square) = \{\rightarrow\rightarrow\rightarrow,\ \rightarrow\leftrightarrows,\ \rightleftarrows\rightarrow,\ \overset{\rightleftarrows}{\leftarrow},\ \rightleftarrows\leftarrow,\ \leftarrow\leftarrow\leftarrow\}$$

and

$$\begin{array}{c} \rightarrow\rightarrow\rightarrow \\ \tilde f \downarrow \\ \rightleftarrows\rightarrow \\ \tilde f \downarrow \\ \rightleftarrows\leftarrow \\ \tilde f \downarrow \\ \leftarrow\leftarrow\leftarrow \end{array} \quad \sqcup \quad \begin{array}{c} \uparrow \tilde e \\ \rightarrow\leftrightarrows \\ \tilde f \downarrow \\ \overset{\rightleftarrows}{\leftarrow} \\ \tilde f \downarrow_{0} \end{array} \qquad \text{so that}$$

$$B(\square\square)\otimes B(\square) \simeq B(\square\square\square) \sqcup B(\square)$$

where

$$B(\square\square\square) = \{\rightarrow\rightarrow\rightarrow,\ \rightleftarrows\rightarrow,\ \rightleftarrows\leftarrow,\ \leftarrow\leftarrow\leftarrow\}.$$

$B(\emptyset)\otimes B(\square) = \{\ ,\ \}$ with

$\tilde{f} \downarrow\uparrow \tilde{e}$ so that $B(\emptyset)\otimes B(\square) \simeq B(\square)$.

An sl_2-<u>crystal</u> is a subset of $B^{\otimes k}$ closed under the action of $\tilde{e}$ and $\tilde{f}$.

The <u>crystal graph</u> of a crystal B has vertices B and edges $b \to b'$ if $\tilde{f}b = b'$.

A crystal is <u>irreducible</u> if the crystal graph is connected.

The <u>character</u> of a crystal B is

$$\mathrm{char}(B) = \sum_{p\in B} x^{wt(p)},$$ where

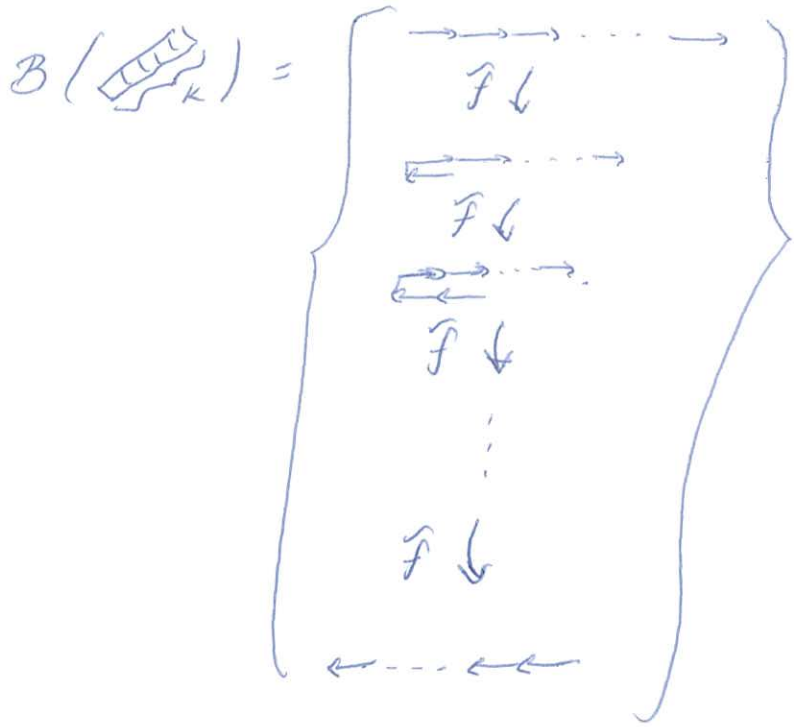
$wt(p)$ is the endpoint of ~~B~~ p.

A <u>highest weight</u> is a path which is always ≥ 0.

A path is highest weight if and only if $\tilde{e}p = 0$.

Theorem

(a) The irreducible $\mathfrak{sl}_2$-crystals are



with

$\mathrm{char}(B(\ \ {}_k))$

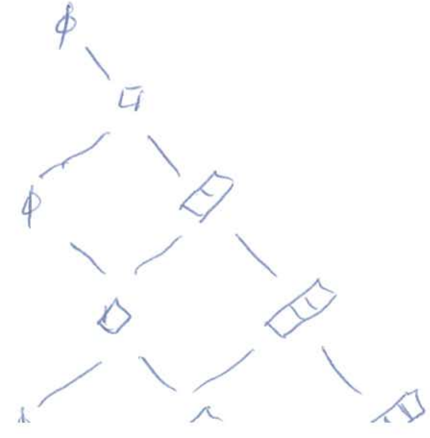
$= x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k}$.

(b) Every crystal is a disjoint union of irreducible crystals.

(c) Every irreducible crystal B has a unique highest weight path p and

$$B \simeq B(\ \ {}_k) \quad \text{if } p \text{ ends at } k.$$

Note that



describes $B(\diamond)^{\otimes k}$ and its decomposition.