Representation Theory 05.12.2009

Equivalences

$$\left\{\begin{array}{l}\text{connected reductive}\\ \text{algebraic groups}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{connected compact}\\ \text{Lie groups}\end{array}\right\}$$

$$G \longmapsto K$$

$$GL_1(\mathbb{C}) = \mathbb{C}^\times \longmapsto S^1 = U(1),$$

where
$$U(1) = S^1 = \{ z \in \mathbb{C}^\times \mid z\bar{z} = 1 \}$$

A *torus* in a compact Lie group is a subgroup isomorphic to $S^1 \times S^1 \times \cdots \times S^1$

A *torus* in an algebraic group is a subgroup isomorphic to $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$.

$$\begin{array}{ccccc} & G & \longrightarrow & K & \\ & \cup\!\mid & & \cup\!\mid & \\ \text{maximal torus in } G & T & \longrightarrow & T_K & \text{maximal torus in } K. \end{array}$$

Next

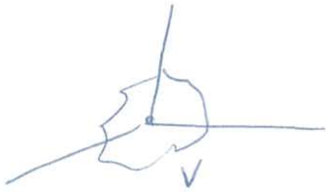
$$\{\text{Lie groups}\} \longleftrightarrow \{\text{Lie algebras}\}$$

$$G \longmapsto \mathfrak{g}$$

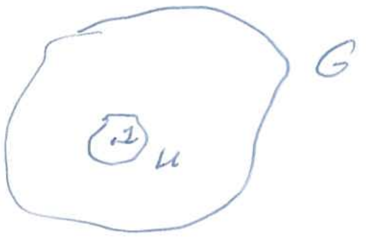
The exponential map

A Lie group is a group that is also a manifold i.e. a topological group that is locally isomorphic to $\mathbb{R}^n$

$$\varphi : V \xrightarrow{\sim} U$$



V is an open neighborhood of O in $\mathbb{R}^n$



U is an open neighborhood of 1 in G

The exponential map is a smooth homomorphism

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ 0 &\longmapsto 1 \\ tX &\longmapsto e^{tX} = \gamma_X(t) \end{aligned} \qquad \text{where } \gamma_X : \mathbb{R} \longrightarrow G$$

is a one parameter subgroup of G, i.e. a smooth group homomorphism $\gamma : \mathbb{R} \longrightarrow G$

The map exp is a homeomorphism on a neighborhood U of O in $\mathfrak{g}$. If G is connected then G is generated by the elements of U.

There are equivalences

$$\left\{\begin{matrix}\text{one parameter} \\ \text{subgroups in } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{tangent vectors} \\ \text{to } 1 \text{ in } \mathfrak{g}\end{matrix}\right\}$$

$$\gamma \longmapsto \gamma_1$$

and

$$\left\{\begin{matrix}\text{left invariant} \\ \text{vector fields}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{tangent vectors} \\ \text{to } 1 \text{ in } \mathfrak{g}\end{matrix}\right\}$$

$$\xi \longmapsto \xi_1, \quad \text{where } \xi_1(f) = (\xi(f))(1)$$

Let G be a Lie group. The <u>ring of functions</u> on G is

$$C^\infty(G) = \{f : G \to \mathbb{R} \mid f \text{ is smooth at } g \text{ for all } g \in G\}$$

where

f is <u>smooth at</u> g if $\left.\dfrac{d^k f}{dx^k}\right|_{x=g}$ exists for all $k \in \mathbb{Z}_{>0}$

Let $g \in G$. A <u>tangent vector</u> to G at g is a linear map $\eta : C^\infty(G) \to \mathbb{R}$ such that

$$\eta(f_1 f_2) = f_1(g)\eta(f_2) + \eta(f_1) f_2(g)$$

for all $f_1, f_2 \in C^\infty(G)$.

A *vector field* is a linear map $\partial : C^\infty(G) \to C^\infty(G)$ such that

$$\partial(f_1 f_2) = f_1 \partial(f_2) + \partial(f_1) f_2,$$

for $f_1, f_2 \in C^\infty(G)$. A *left invariant vector field* on G is a vector field $\partial : C^\infty(G) \to C^\infty(G)$ such that

$$L_g \xi = \xi L_g \quad \text{for all } g \in G,$$

where $L_g : C^\infty(G) \to C^\infty(G)$ is given by

$$(L_g f)(x) = f(g^{-1}x), \quad \text{for } f \in C^\infty(G),\ g, x \in G.$$

The *Lie algebra* of G is the vector space $\mathfrak{g}$ of left invariant vector fields on G with bracket

$$[\partial_1, \partial_2] = \partial_1 \partial_2 - \partial_2 \partial_1.$$

Let $\gamma : \mathbb{R} \to G$ be a one parameter subgroup of G. Define

$$\frac{df(\gamma(t))}{dt} = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$$

and let $\gamma_1 : C^\infty(G) \to \mathbb{R}$ be the tangent vector at 1 given by

$$\gamma_1(f) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0}.$$

<u>Example</u> The Lie algebra $\mathfrak{gl}_n$ is

$\mathfrak{gl}_n = \{x \in M_n(\mathbb{C})\}$ with bracket

$$[x_1, x_2] = x_1 x_2 - x_2 x_1.$$

Our favorite basis of $\mathfrak{gl}_n$ is

$$\{E_{ij} \mid 1 \le i, j \le n\}.$$

The exponential map is

$$\begin{aligned} \mathfrak{gl}_n &\longrightarrow GL_n \\ tX &\longmapsto e^{tX} \end{aligned} \quad \text{where } e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

for a matrix A. Then

$$e^{tE_{ij}} = 1 + tE_{ij} = \begin{pmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad \text{for } i \ne j. \text{ and}$$

$$e^{tE_{ii}} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^t & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = h_i(e^t).$$

If $n=1$ the exponential map

$$\begin{aligned} e : \mathbb{C} &\longrightarrow \mathbb{C}^\times \\ tx &\longmapsto e^{tx} \end{aligned}$$ is a homeomorphism from

a neighborhood of 0 to a neighborhood of 1. In fact

$$e(s+t) = e(s)\, e(t) \quad \text{forces}$$

$$e(t) = 1 + a_1 t + a_1^2 t^2/_{2!} + a_1^3 t^3/_{3!} + \cdots \quad \text{for some } a_1 \in \mathbb{C}.$$