Representation Theory 13.05.2009

A morphism $f: X \to Y$ provides

$$df : T_x(X) \longrightarrow T_{f(x)}(Y), \quad \text{for } x \in X.$$

Let G be an algebraic group.

Let M be a G-module,

$$\begin{aligned} \rho : G &\longrightarrow GL(M) \\ g &\longmapsto \rho(g) \\ e^{tx} &\longmapsto \rho(e^{tx}), \end{aligned}$$

the corresponding representation of G. If

$$d\rho(x) = \frac{d}{dt}\rho(e^{tx})\Big|_{t=0} \quad \text{then } \rho(e^{tx}) = e^{t\,d\rho(x)}$$

and we get a representation of $\mathfrak{g}$ on M

$$\begin{aligned} d\rho : \mathfrak{g} &\longrightarrow \operatorname{End}(M) \\ x &\longmapsto d\rho(x) \end{aligned}$$

Let G be an algebraic group.

The <u>conjugation action</u> of G on G is given by

$$\begin{aligned} \mathrm{Int}_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1}, \end{aligned} \qquad \text{for } g \in G$$

The differential of these maps gives the <u>Adjoint action</u> of G on $\mathfrak{g}$

$$\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \text{for } g \in G.$$

Then $\mathfrak{g}$ acts on $\mathfrak{g}$ by the <u>adjoint action</u>

$$\begin{aligned} \mathrm{ad}_y : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x &\longmapsto [y,x] \end{aligned} \qquad \text{for } y \in \mathfrak{g}$$

since

$$\mathrm{Ad}_{e^{ty}}(x) = e^{ty} x e^{-ty} = \left(1 + ty + \frac{t^2 y^2}{2!} + \cdots\right) x \left(1 - ty + \frac{t^2 y^2}{2!} - \frac{t^3 y^3}{3!} + \cdots\right)$$

$$= x + t(yx - xy) + \frac{t^2}{2}(y^2 x - 2yxy + xy^2) + \cdots$$

$$= (e^{t\,\mathrm{ad}_y})(x),$$

Note: $(\mathrm{ad}_y)^2 = [y,[y,x]] = [y, yx - xy] = y^2x - yxy - yxy + xy^2.$

Let G be an algebraic group. A <u>torus</u> in G is a subgroup H isomorphic to $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$.

Let K be a Lie group. A <u>torus</u> in K is a subgroup T isomorphic to $S^1 \times \cdots \times S^1$.

Let $\mathfrak{g}$ be a Lie algebra.

An <u>abelian Lie subalgebra</u> is a Lie subalgebra $\mathfrak{h}$ such that

$$[h_1, h_2] = 0, \quad \text{for } h_1, h_2 \in \mathfrak{h}.$$

A <u>Cartan subalgebra</u> is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

<u>Example</u>: A maximal torus of $GL_n(\mathbb{C})$ is

$$H = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \;\middle|\; x_1, \ldots, x_n \in \mathbb{C}^\times \right\}$$

A Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} \;\middle|\; h_1, \ldots, h_n \in \mathbb{C} \right\}.$$

Note that $\mathfrak{h} = \mathrm{Lie}(H) = T_1(H)$.

The irreducible (rational) representations of H are

$$X^\mu = X^{\mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n} = X^{\mu_1\varepsilon_1}\cdots X^{\mu_n\varepsilon_n}$$
$$= (X^{\varepsilon_1})^{\mu_1}\cdots(X^{\varepsilon_n})^{\mu_n}, \text{ with } \mu_1,\ldots,\mu_n\in\mathbb{Z}$$

where

$$X^{\varepsilon_i}: H \longrightarrow \mathbb{C}^\times$$
$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n\end{pmatrix} \longmapsto x_i$$

The irreducible representations of $\mathfrak{h}$ are

$$\mu: \mathfrak{h} \to \mathbb{C}, \text{ so that } \mu\in\mathfrak{h}^*,$$

and $\mu = \mu_1\varepsilon_1+\cdots+\mu_n\varepsilon_n$, with $\mu_1,\ldots,\mu_n\in\mathbb{C}$ and

$$\varepsilon_i: \mathfrak{h} \longrightarrow \mathbb{C}$$
$$\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n\end{pmatrix} \longmapsto h_i.$$

Hence $\mathfrak{h}^*$ indexes irreducible representations of $\mathfrak{h}$

$\{X^\mu \mid \mu\in\mathfrak{h}^*_{\mathbb{Z}}\}$ are the irreducible reps of H.

Weights

Let M be a G-module and

$X^\mu : H \to \mathbb{C}^\times$ an irreducible representation of H.

The *μ-weight space* of M is

$$M_\mu = \{ m \in M \mid \text{for each } t \in H,\ tm = X^\mu(t) m \}$$

$$= \{ m \in M \mid \text{for each } h \in \mathfrak{h},\ hm = \mu(h) m \}$$

The *generalized μ-weight space* of M is

$$M_\mu^{gen} = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{for each } t \in H \\ (t - X^\mu(t))^\ell m = 0 \text{ for some } \ell \in \mathbb{Z}_{>0} \end{array} \right\}$$

$$= \left\{ m \in M \;\middle|\; \begin{array}{l} \text{for each } h \in \mathfrak{h} \\ (h - \mu(h))^\ell m = 0 \text{ for some } \ell \in \mathbb{Z}_{>0} \end{array} \right\}$$

Then $M_\mu \subseteq M_\mu^{gen}$ and $M_\mu^{gen} \neq 0$ implies $M_\mu \neq 0$.

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{gen}$$

The *weights* of M are the μ such that $M_\mu \neq 0$.