Representation Theory 17.03.2009.

The algebra $A = M_d(\mathbb{C})$

A acts on $\mathbb{C}^d = \mathrm{span}\{e_1, \dots, e_n\}$ with $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith

by left multiplication.

Theorem

(a) If M is a simple $M_d(\mathbb{C})$-module then $M \cong \mathbb{C}^d$

(b) If $\vec{t} : M_d(\mathbb{C}) \to \mathbb{C}$ is a trace then

$$\vec{t} = \kappa \cdot Tr, \text{ for some } \kappa \in \mathbb{C}.$$

(c) $Z(M_d(\mathbb{C})) = \mathbb{C} \cdot Id$

(d) $M_d(\mathbb{C})$ has a unique nonzero ideal.

Our favorite basis of $M_d(\mathbb{C})$ is $\{E_{ij} \mid 1 \le i, j \le d\}$
with
$$E_{ij} E_{kl} = \delta_{jk} E_{il}.$$

Note that $E_{ij} e_k = \delta_{jk} e_i$.

Proof of (a) Let M be a simple module.

Let $m \in M$, $m \neq 0$. Then

$$0 \neq m = 1 \cdot m = \sum_{i=1}^{d} E_{ii} m, \text{ so some } E_{ii} m \neq 0.$$

Claim: $\begin{array}{l} \mathbb{C}^d \to M \\ e_i \mapsto E_{ii} m \\ e_j \mapsto E_{ji} m \end{array}$ is an isomorphism.

This will follow from Schur's lemma if $\mathbb{C}^d$ is a simple module.

<u>The algebra</u> $A = \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C})$

A has basis $\{E_{ij}^\lambda \mid \lambda\in\hat{A},\ 1\le i,j\le d_\lambda\}$ with

$$E_{ij}^\lambda E_{rs}^\mu = \delta_{\lambda\mu}\delta_{jr}E_{is}.$$

Define $A^\lambda = \mathrm{span}\{e_i^\lambda \mid 1\le i\le d_\lambda\}$ with

$$E_{ij}^\mu e_r^\lambda = \delta_{\lambda\mu}\delta_{jr}e_i^\lambda.$$

Define $Tr^\lambda : A\to\mathbb{C}$ by $Tr^\lambda(E_{ij}^\mu) = \delta_{\lambda\mu}\delta_{ij}$.

Define $z_\lambda = \sum_{i=1}^{d_\lambda} E_{ii}^\lambda$ so that $z_\lambda^2 = z_\lambda$ and $z_\lambda z_\mu = 0$

Define $I^\lambda = z_\lambda\cdot A = \mathrm{span}\{E_{ij}^\lambda \mid 1\le i,j\le \lambda\}$.

<u>Theorem</u>

(1) A^λ, $\lambda\in\hat{A}$, are the simple A-modules

(2) If $\vec{t} : A\to\mathbb{C}$ is a trace then

$$\vec{t} = \sum_{\lambda\in\hat{A}} t_\lambda Tr^\lambda, \quad \text{with } t_\lambda\in\mathbb{C}.$$

(3) $Z(A) = \mathrm{span}\{z_\lambda \mid \lambda\in\hat{A}\}$

(4) The minimal ideals of A are I^λ, $\lambda\in\hat{A}$.
The ideals of A are sums of I^λ.

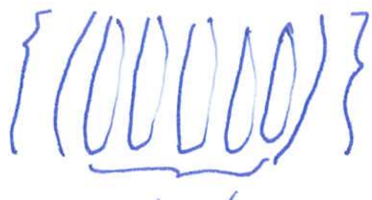
Recall:

Theorem (Artin-Wedderburn/Peter-Weyl).
Let A be an algebra such that the trace $\vec{t}_A$ of the regular representation of A is nondegenerate. Then

$$A \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}), \text{ for some } \hat{A} \text{ and some } d_\lambda \in \mathbb{Z}_{\geq 0}$$

Example 1: As A-modules ($A = M_d(\mathbb{C})$)

$$A \cong (\mathbb{C}^d)^{\oplus d} =$$



d columns

So $\vec{t}_A = d \cdot Tr$ (i.e. $\vec{t}_A(E_{ij}) = \sum_{r,s} E_{ij} E_{rs}\big|_{E_{rs}} = \sum_{r,s=1}^{d} \delta_{jr} E_{is}\big|_{E_{rs}}$
$= \delta_{ij} \cdot d$).

Example 2: If $A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$ then, as A-modules

$$A \cong \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda} \quad \text{and} \quad \vec{t}_A = \sum_{\lambda \in \hat{A}} d_\lambda Tr^\lambda.$$

The dual basis to $\{E_{ij}^\lambda \mid \lambda \in \hat{A}, 1 \leq i, j \leq \lambda\}$ is

$$\{\tfrac{1}{d_\lambda} E_{ji}^\lambda \mid \lambda \in \hat{A}, 1 \leq i, j \leq \lambda\}.$$

If $\begin{array}{l} A \longrightarrow End(A^\lambda) = M_{d_\lambda}(\mathbb{C}) \\ a \longmapsto A^\lambda(a) \end{array}$ are the irreducible representations of A.

then $A^\lambda(E_{ij}^\mu) = \delta_{\lambda\mu} E_{ij}^\lambda$ and

$$E_{ij}^\mu = \sum_{\substack{\lambda \in \hat{A} \\ 1 \le r,s \le d_\lambda}} d_\lambda \, A^\lambda\left(\frac{1}{d_\lambda} E_{sr}^\mu\right)_{ji} E_{rs}^\mu .$$

Let B be another basis of A and

B^* the dual basis with respect to $\langle , \rangle$,

where $\langle a_1, a_2 \rangle = \vec{t}_A(a_1 a_2)$. Then

$$E_{ij}^\mu = \sum_{b \in B} d_\lambda \, A^\lambda(b^*)_{ji} \, b \qquad \text{and}$$

$$b = \sum_{\substack{\lambda \in \hat{A} \\ 1 \le i,j \le d_\lambda}} A^\lambda(b)_{ij} \, E_{ij}^\lambda .$$

These are the <u>Fourier</u> <u>Inversion</u> <u>Formulas</u>.

<u>Pullback</u> <u>functors</u> Suppose $A \xrightarrow{\varphi} R$ is an algebra homomorphism. Then we get a functor

$$\begin{array}{ccc} \{R\text{-modules}\} & \xrightarrow{\varphi^*} & \{A\text{-modules}\} \\ M & \longmapsto & \varphi^*(M) \end{array}$$

where $\varphi^*(M) = M$ as vector spaces and the A-action is given by $a \cdot m = \varphi(a) m$, for $a \in A$, $m \in M$

If φ is injective

$$A \hookrightarrow R,$$ then A is a subalgebra of R

and

$$\varphi^*(M) = \mathrm{Res}_A^R(M)$$ is the <u>Restriction</u> <u>functor</u>

If φ is surjective,

$$A \twoheadrightarrow R$$ then

if M is a simple R-module, then $\varphi^*(M)$ is a simple A-module.