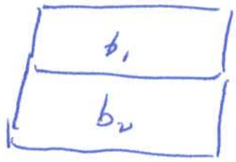
Representation Theory 18.03.2009

The braid group B_k

B_k is the group of braids on k strands with product

$b_1 b_2 =$ 

Theorem (Artin) B_k is presented by generators

$$T_i = \overset{1 \;\cdots\; i \;\; i+1 \;\cdots\; k}{|||| \times ||||}, \quad 1 \le i \le k-1$$

with relations

$$T_i T_j = T_j T_i \text{ if } j \ne i, i \pm 1, \quad \text{and} \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

The symmetric group S_k is the quotient of B_k by the relations

$$s_i^2 = 1,$$

Write $s_i = \overset{i \;\, i+1}{||||X|||}$ since $g_i = |||| \times |||| = g_i^{-1} = |||| \times ||||$ in S_k.

The Iwahori-Hecke algebra is the quotient of $\mathbb{C}B_k$ by the relations

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$$

Let $e_i = t^{\frac{1}{2}} - T_i$ in H_n.

Then $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ becomes

$$e_i e_{i+1} e_i - e_{i+1} e_i e_{i+1} = e_i - e_{i+1}$$

The map

$$\begin{aligned} \mathbb{C}B_k &\longrightarrow H_k \longrightarrow TL_k \\ T_i &\longmapsto T_i \\ & \qquad e_i \longmapsto e_i \end{aligned}$$

are surjective algebra homomorphisms.

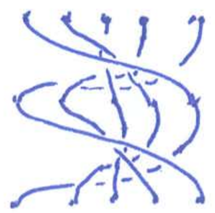
The maps

$$\begin{aligned} B_k &\longrightarrow B_{k+1} \\ b &\longmapsto \boxed{\quad b \quad}\ | \end{aligned}$$

give inclusions

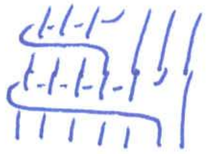
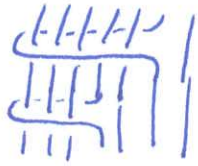
$$\begin{matrix} H_1 \subseteq H_2 \subseteq \cdots \\ TL_1 \subseteq TL_2 \subseteq \cdots \end{matrix} \quad \text{and} \quad \mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq \cdots$$

The element

$$Z_k = T_{w_0}^2 =$$  $$\in Z(B_k)$$

and $T_{w_0}^2 = y^{\varepsilon_1} \cdots y^{\varepsilon_k}$ where $y^{\varepsilon_i} =$ 

and

$y^{\varepsilon_i} y^{\varepsilon_j} =$  $=$  $= y^{\varepsilon_j} y^{\varepsilon_i}$

So the image of $\mathbb{C}[y^{\varepsilon_1}, \ldots, y^{\varepsilon_k}]$ is a large commutative subalgebra of H_k (or TL_k).

Note that $\mathbb{C}[Z_1^{\pm 1}, Z_2^{\pm 1}, \ldots, Z_k^{\pm 1}]$ is commutative since $Z_i Z_k = Z_k Z_i$ for $i \le k$

and $\mathbb{C}[Z_1^{\pm 1}, \ldots, Z_k^{\pm 1}] = \mathbb{C}[y^{\pm\varepsilon_1}, \ldots, y^{\pm\varepsilon_k}]$ since $y^{\varepsilon_i} = Z_i Z_{i-1}^{-1}$.

Bratelli diagrams

The <u>Bratelli diagram</u> for $A \subseteq B$ has

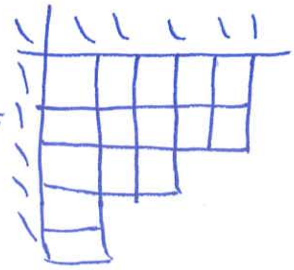
(a) vertices $\hat{A}$ on level A

vertices $\hat{B}$ on level B

(b) There are $m_{\lambda\mu}$ edges from $\lambda \to \mu$ if

$$\mathrm{Res}^B_A (B^\mu) \cong \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus m_{\lambda\mu}}.$$

A <u>partition</u> is a collection of boxes in a corner.

$\lambda =$  $= (5,5,3,1,1)$

The Bratelli diagram for

$$TL_1 \subseteq TL_2 \subseteq TL_3 \subseteq \cdots$$

has (a) partitions with k boxes ≤ 2 rows on level k

(b) an edge $\lambda - \mu$ if μ is obtained from λ by adding a box.

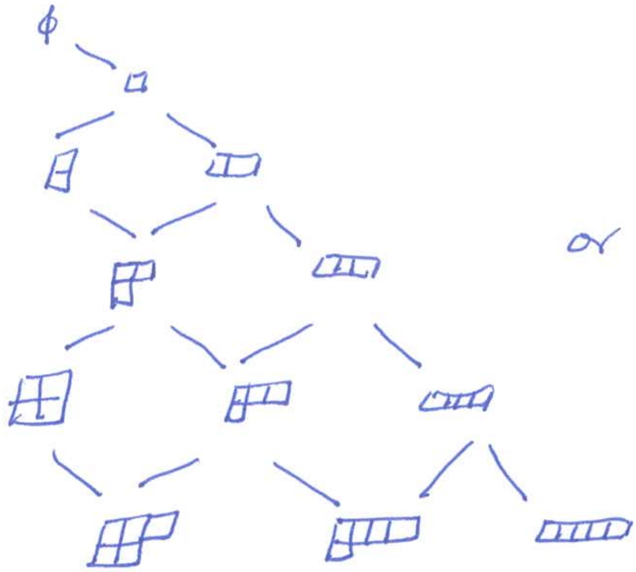
The Bratelli diagram for

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$$

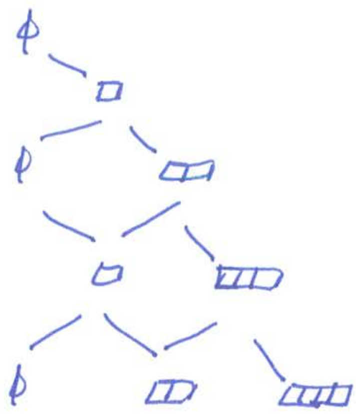
has (a) partitions with k boxes on level k.

(b) an edge $\lambda - \mu$ if μ is obtained from λ by adding a box.

For $TL_1 \subseteq TL_2 \subseteq TL_3 \subseteq \dots$



or

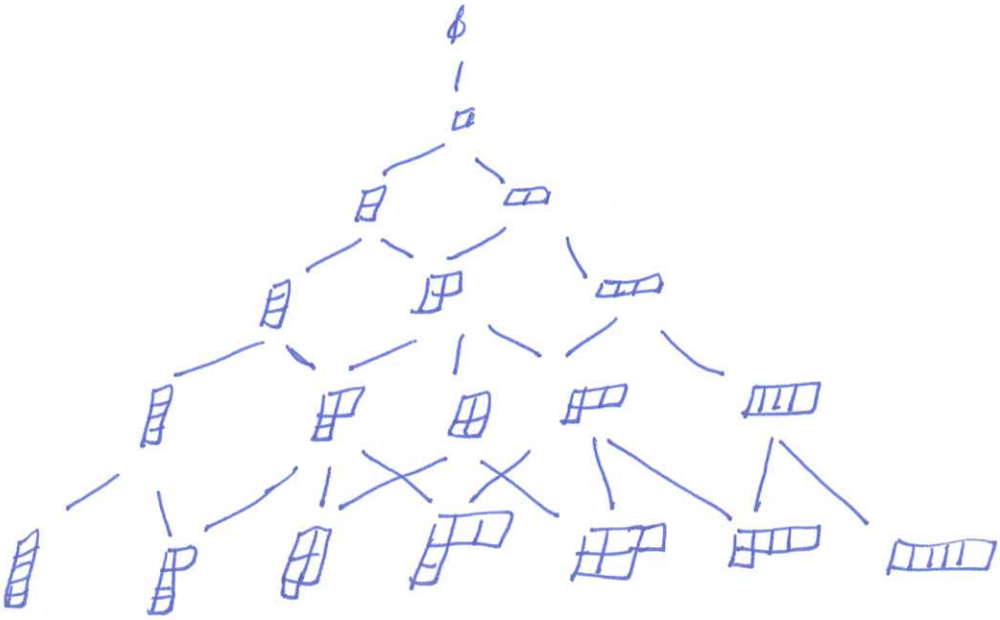


with

(a) partitions with ≤ 1 row and $k, k-2, k-4, \dots$ boxes on level k,

(b) an edge $\lambda - \mu$ if μ is obtained by adding or removing a box.

For $H_1 \subseteq H_2 \subseteq \dots$

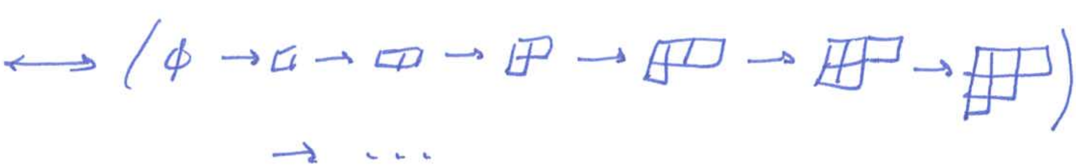


A <u>standard</u> <u>tableau</u> of shape λ is a filling of the boxes of λ with $1, 2, \ldots, k$ such that

(a) rows increase left to right

(b) columns increase top to bottom.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & 9 \\ \hline 6 & 10 \\ \cline{1-2} 11 \\ \cline{1-1}\end{array} \longleftrightarrow$$



$$\left\{\begin{matrix}\text{standard tableaux} \\ \text{of shape } \lambda\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{paths } \phi \to \cdots \to \lambda \\ \text{in } \hat{H}\end{matrix}\right\}.$$