Representation Theory 20.05.2009.

Let

$$(V\otimes V)^* = \{\text{bilinear forms on } V\}$$

$$S^2(V)^* = \{\text{symmetric bilinear forms on } V\}$$

$$\Lambda^2(V)^* = \{\text{skew symmetric bilinear forms on } V\}.$$

A choice of basis $b_1, \dots, b_n$ of V provides a bijection

$$(V\otimes V)^* \longrightarrow M_n(\mathbb{F})$$

$$\langle\ \rangle \longmapsto A = (\langle b_i, b_j\rangle)_{1\le i,j\le n}$$

and this bijection identifies

$S^2(V)^*$ with $Sym_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid A = A^t\}$

$\Lambda^2(V)^*$ with $Skew_n(\mathbb{F}) = \{A\in M_n(\mathbb{F}) \mid A = -A^t\}$

The S_2-action and the $GL(V)$ action on $(V\otimes V)^*$ become the actions of S_2 and $GL_n(\mathbb{F})$ on $M_n(\mathbb{F})$ given by

$$s\cdot A = A^t \quad \text{and} \quad g\cdot A = g^{-1}A(g^{-1})^t$$

for $A\in M_n(\mathbb{F})$ and $g \in GL_n(\mathbb{F})$.

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(1) Under the action of $GL_n(\mathbb{F})$ on the set $Skew_n(\mathbb{F}) = \{ A \in M_n(\mathbb{F}) \mid A = -A^t \}$ the orbits have representatives

$$J_r = \left(\begin{array}{c|c|c} \overbrace{\quad 0 \quad}^{r} & \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & 0 \\ \hline \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \quad \text{for} \quad 0 \le r \le \left\lfloor \frac{n}{2} \right\rfloor$$

(2) Under the action of $GL_n(\mathbb{F})$ on the set $Herm_n(\mathbb{F}) = \{ A \in M_n(\mathbb{F}) \mid A = +\bar{A}^t \}$ the orbits have representatives

$$\begin{pmatrix} \alpha_1 & & & & & & \\ & \alpha_2 & & & & & \\ & & \ddots & & & & \\ & & & \alpha_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \quad \text{with} \quad \alpha_i = \bar{\alpha_i} \neq 0.$$

Note that if $^{-} : \mathbb{F} \to \mathbb{F}$ is the identity then $Herm_n(\mathbb{F}) = Sym_n(\mathbb{F})$.

Proof Choose a basis of V, $b_1, \ldots, b_n$ such that for each ℓ the form $\langle\ \rangle$ is nondegenerate on $\operatorname{span}\{b_1, \ldots, b_\ell\}$.

Let
$$D_{j\ell} = \operatorname{cof}\left(\left(\langle b_i, b_j\rangle\right)_{1\le i,j\le \ell}\right)_{j\ell}$$

and
$$e_\ell = D_{\ell\ell}^{-1}\left(D_{1\ell}b_1 + D_{2\ell}b_2 + \cdots + D_{\ell\ell}b_\ell\right).$$

Then
$$\langle e_i, e_j\rangle = 0, \text{ if } i \neq j \text{ and}$$
$$\langle e_\ell, e_\ell\rangle = D_{\ell\ell}^{-1} D_{\ell+1,\ell+1}$$

If $P = \begin{pmatrix} & & \\ & D_{\ell\ell}^{-1}D_{i\ell} & \\ & & \end{pmatrix}$ then $P^t A P = \begin{pmatrix} D_{11}^{-1}D_{22} & & 0 \\ & D_{22}^{-1}D_{33} & \\ 0 & & \ddots \end{pmatrix}$

HW: Do an example.

Note: This is the same as Gram-Schmidt.

Two more G-actions

(1) $GL(V) \times GL(W)$ acts on $V \otimes W^*$.

This corresponds to the action of

$GL_n(\mathbb{F}) \times GL_m(\mathbb{F})$ on $M_{n \times m}(\mathbb{F})$ by

$$(P, Q) \cdot A = PAQ^{-1}.$$

(2) $GL(V)$ acts on $V \otimes V^*$

This corresponds to the action of

$GL_n(\mathbb{F})$ on $M_n(\mathbb{F})$ by $P \cdot A = PAP^{-1}$.

In this case the orbits are indexed by Jordan forms.

The answer coincides with the classification of finitely generated $\mathbb{F}[x]$-modules:

$$: \mathbb{F}[x] \longrightarrow M_n(\mathbb{F}) = End(V)$$
$$x \longmapsto A$$

i.e. representations V of $\mathbb{F}[x]$ of dimension n.

Smith normal form

If A is an $m \times n$ matrix over a principal ideal domain R then there are

$P \in GL_m(R)$ and $Q \in GL_n(R)$

such that

$$PAQ = \begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & & & \\ & & \ddots & & & & \\ & & & a_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

where $a_i \neq 0$ and $a_1 | a_2 | \cdots | a_r$.

This is accomplished by row and column reduction with elementary matrices

$$h_i(c) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \qquad x_{ij}(c) = \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$s_{ij} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & 1 & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & 1 & & & 0 & \\ & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix} \quad \underline{\text{and}} \quad y_{i,i+1}(u,v,s,t) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & u & s & & \\ & & & v & t & & \\ & & & & & 1 & \\ & 0 & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}$$

References

In

P.B. Bhattacharya, S.K. Jain and S.R. Nagpaul, Basic Abstract Algebra, 2nd Edition, Cambridge University Press 1994,

the classification of the $GL(V) \times GL(W)$ orbits on $V \otimes W^*$ is termed Smith Normal Form and is proved, over a PID, by row reduction, in Theorem 3.2 of Chapter 20. Invariant factors are defined on p. 399.

In

M. Artin, Algebra, Prentice Hall, 1991,

this appears in Theorem 4.3 of Chapter 12. Again the proof is by row reduction. Artin Chapter 7 is on bilinear forms.

The classification of $GL_n(\mathbb{R})$ orbits on $Sym_{n \times n}(\mathbb{R})$ is in Theorem (2.9) of Chapter 7 and is termed Sylvester's law. The proof is the Gram-Schmidt induction

The classification of $U_n(\mathbb{C})$ orbits on $Herm_{n \times n}(\mathbb{C})$ is done in Theorem (5.4), finding a basis of eigenvectors. This is termed, the Spectral Theorem.

In

N. Bourbaki, Algèbre, Chapt. 9, Forme sesquilinéaires et formes quadratiques,

the $GL_n(\mathbb{F})$ orbits of $Herm_{n\times n}(\mathbb{F})$ are classified in Corollary 2 to Theorem 1 of §6 no 1 and the explicit version of the Gram-Schmidt process is in Proposition 1 of §6 no. 1, as noted in the sentence immediately before Proposition 2 on p. 93. As noted there, the Gram-Schmidt process works perfectly well over commutative rings.

In Artin, the Jordan normal form ($GL_n(\mathbb{F})$ orbits on $GL_n(\mathbb{F})$ by conjugation) is derived from the Smith normal form applied to the ring $\mathbb{F}[t]$. This is Theorem 7.13. The rational canonical form appears in Theorem 7.9.

In Bhattacharya-Jain-Nagpaul the rational canonical form is in §4 of Chapter 21 and the Jordan canonical form is in §5 of Chapter 21.

In N. Bourbaki, Algèbre, Chapt VII the Jordan canonical form is Proposition 4 of §5 No 3 and proposition 8 of §5 No 4. The rational canonical form is in §5 No 1 and the Smith normal form is Cor. 1 of §4. No. 6. There is a thorough discussion of invariant factors and elementary divisors.