Representation Theory 31.03.2009

The enveloping algebra $U\mathfrak{sl}_2$ has generators $e, f, h$ and relations

$$ef = fe + h, \quad hf = fh - 2f, \quad eh = he - 2e.$$

$U\mathfrak{sl}_2$ has basis $\{ f^{n_1} h^{n_2} e^{n_3} \mid n_1, n_2, n_3 \in \mathbb{Z}_{\geq 0} \}$.

Our favorite representation of $U\mathfrak{sl}_2$ is

$$\begin{aligned} U\mathfrak{sl}_2 &\longrightarrow M_2(\mathbb{C}) \\ e &\longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &\longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

i.e. $L(\square) = L(1) = \mathrm{span}\{v_1, v_{-1}\}$

with
$$\begin{aligned} ev_1 &= 0, & hv_1 &= v_1, & fv_1 &= v_{-1} \\ ev_{-1} &= v_1, & hv_{-1} &= v_{-1}, & fv_{-1} &= 0 \end{aligned}$$

Another favourite representation is

$$\begin{aligned} \varepsilon : U\mathfrak{sl}_2 &\longrightarrow M_1(\mathbb{C}) \\ e &\longmapsto 0 \\ f &\longmapsto 0 \\ h &\longmapsto 0 \\ 1 &\longmapsto 1 \end{aligned}$$

i.e. $L(\emptyset) = L(0) = \mathrm{span}\{v_0\}$

with $ev_0 = 0$,
$fv_0 = 0$,
$hv_0 = 0$.

A Hopf algebra is an algebra $A$ with a map $\Delta: A \to A\otimes A$ such that if $M$ and $N$ are $A$-modules,

$$M = \mathrm{span}\{m_1, \ldots, m_r\} \text{ and } N = \mathrm{span}\{n_1, \ldots, n_s\}$$

then $M \otimes N = \mathrm{span}\{m_i \otimes n_j \mid 1 \le i \le r,\ 1 \le j \le s\}$ with $A$-action given by

$$a(m\otimes n) = \sum_a a_{(1)} m \otimes a_{(2)} n, \quad \text{if } \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$$

is an $A$-module.

$U\mathfrak{sl}_2$ is a Hopf algebra with

$$\Delta(e) = e\otimes 1 + 1 \otimes e, \quad \Delta(f) = f\otimes 1 + 1\otimes f, \quad \Delta(h) = 1\otimes h + h \otimes 1.$$

Hence $L(\Delta)\otimes L(\Delta) = \mathrm{span}\{v_1\otimes v_1, v_1 \otimes v_{-1}, v_{-1}\otimes v_1, v_{-1}\otimes v_{-1}\}$ is an $U\mathfrak{sl}_2$ module with

$$e(v_1 \otimes v_1) = 0$$

$$f(v_1\otimes v_1) = v_{-1}\otimes v_1 + v_1 \otimes v_{-1}$$

$$f^2(v_1\otimes v_1) = 0 + v_{-1}\otimes v_{-1} + v_{-1}\otimes v_{-1} + 0 = 2v_{-1}\otimes v_{-1}$$

$$f^3(v_1 \otimes v_1) = 0.$$

Let

$$b_2 = v_1\otimes v_1, \quad b_0 = v_{-1}\otimes v_1 + v_1 \otimes v_{-1}, \quad b_{-2} = v_{-1}\otimes v_{-1}$$

$$b_0' = v_1 \otimes v_{-1} - v_{-1} \otimes v_1 .$$

Then

$$e(v_1 \otimes v_{-1} - v_{-1} \otimes v_1) = v_1 \otimes v_1 - v_1 \otimes v_{+1} = 0 \quad \text{and}$$

$$f(v_1 \otimes v_{-1} - v_{-1} \otimes v_1) = v_{-1} \otimes v_{-1} - v_{-1} \otimes v_{-1} = 0$$

In the basis $L(\Box) \otimes L(\Box) = \text{span}\{b_2, b_0, b_{-2}, b_0'\}$

$$e \longmapsto \left(\begin{array}{ccc|c} 0 & 2 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ \hline & & & 0 \end{array}\right) \qquad h \longmapsto \left(\begin{array}{ccc|c} 2 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & -2 & \\ \hline & & & 0 \end{array}\right)$$

$$f \longmapsto \left(\begin{array}{ccc|c} 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ 0 & 2 & 0 & \\ \hline & & & 0 \end{array}\right)$$

and

$$L(\Box) \otimes L(\Box) \cong L(\Box\!\Box) \oplus L(\emptyset) \quad \text{where}$$

$$L(\Box\!\Box) = \text{span}\{v_2, v_0, v_{-2}\} \quad \text{with}$$

$$\begin{array}{lll} e v_2 = 0, & f v_2 = v_0 & h v_2 = 2v_2 \\ e v_0 = 2v_2, & f v_0 = +2v_{-2} & h v_0 = 0 v_0 \\ e v_{-2} = v_0, & f v_{-2} = 0 & h v_{-2} = -2v_0 \end{array}$$

Next

$$L(\Box) \otimes L(\Box) \otimes L(\Box) = \text{span}\left\{\begin{array}{l} v_1 \otimes v_1 \otimes v_1,\ v_1 \otimes v_1 \otimes v_{-1} \\ v_1 \otimes v_{-1} \otimes v_1,\ v_1 \otimes v_{-1} \otimes v_{-1} \\ v_{-1} \otimes v_1 \otimes v_1,\ v_{-1} \otimes v_1 \otimes v_{-1} \\ v_{-1} \otimes v_{-1} \otimes v_1,\ v_{-1} \otimes v_{-1} \otimes v_{-1} \end{array}\right\}$$

$$= \text{span}\{b_2 \otimes v_1, b_2 \otimes v_{-1}, b_0 \otimes v_1, b_0 \otimes v_{-1}, b_{-2} \otimes v_1, b_{-2} \otimes v_{-1}, b_0' \otimes v_1, b_0' \otimes v_{-1}\}$$

Then let $c_3 = b_2 \otimes v_1$

$$e(b_2 \otimes v_1) = 0$$

$$c_1 = f(b_2 \otimes v_1) = b_0 \otimes v_1 + b_2 \otimes v_{-1}$$

$$2c_{-1} = f^2(b_2 \otimes v_1) = 2b_{-2} \otimes v_1 + b_0 \otimes v_{-1} + b_0 \otimes v_{-1} + 0 = 2b_{-2} \otimes v_1 + 2b_0 \otimes v_{-1}$$

$$2\cdot 3\, c_{-3} = f^3(b_2 \otimes v_1) = 0 + 2b_{-2} \otimes v_{-1} + 2b_{-2} \otimes v_{-1} + 0 + 2b_{-2} \otimes v_{-1} + 0 = 6b_{-2} \otimes v_{-1}$$

$$f^4(b_2 \otimes v_1) = 0.$$

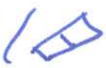
and let $c_1' = b_0 \otimes v_1 - 2b_2 \otimes v_1$

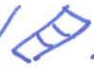
$$e(b_0 \otimes v_1 - 2b_2 \otimes v_{-1}) = 0$$

$$c_{-1}' = f(b_0 \otimes v_1 - 2b_2 \otimes v_{-1}) = 2b_{-2} \otimes v_1 + b_0 \otimes v_{-1} - 2b_0 \otimes v_{-1} - \cancel{2b_2 \otimes v}0$$

$$= 2b_{-2} \otimes v_1 - b_0 \otimes v_{-1}$$

$$f^2(b_0 \otimes v_1 - 2b_2 \otimes v_{-1}) = 0.$$

So $L($  $) \otimes L($  $) \cong L($  $) \oplus L($  $)$

where $L($  $) = \text{span}\{v_3, v_1, v_{-1}, v_{-3}\}$ with

$ev_3 = 0$, $\quad fv_3 = v_1$

$ev_1 = 3v_3$, $\quad fv_1 = 2v_{-1}$

$ev_{-1} = 2v_1$, $\quad fv_{-1} =$

$ev_{-3} = v_{-1}$,

so that

$$h \longmapsto \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix}$$

$$e \longmapsto \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad f \longmapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

and $L(\square)\otimes L(\square)\otimes L(\square) \cong \left(L(\square\square)\oplus L(\phi)\right)\otimes L(\square)$

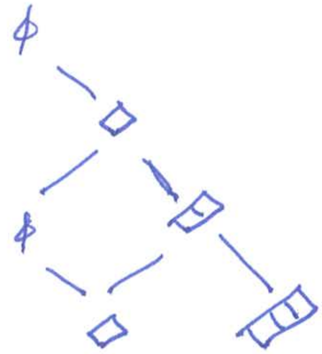
$= L(\square\square)\otimes L(\square) \oplus L(\phi)\otimes L(\square)$

$= \left(L(\square\square\square)\oplus L(\square)\right) \oplus \left(L(\square)\right)$

$L(\square)$

$L(\square)^{\otimes 2}$

$L(\square)^{\otimes 3}$



Is there a connection to $T_L$?