Representation Theory 05.05.2010

<u>Projective space</u> $\mathbb{P}^1$ is the space of lines in $\mathbb{C}^2$

$$\mathbb{P}^1 = \{0 \subseteq [v] \subseteq \mathbb{C}^2 \mid v \in \mathbb{C}^2, v \neq 0\}$$

$$= \frac{\{[z_0, z_1] \mid z_0, z_1 \in \mathbb{C}, (z_0, z_1) \neq 0\}}{([z_0, z_1] = [\lambda z_0, \lambda z_1] \text{ for } \lambda \in \mathbb{C}^*)}.$$

Our favorite point of $\mathbb{P}^1$ is

$[1,0] = \text{span}\{e_1\}$, where $e_1 = (1,0)$

which has stabilizer

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ under the GL_2 action on $\mathbb{C}^2$.

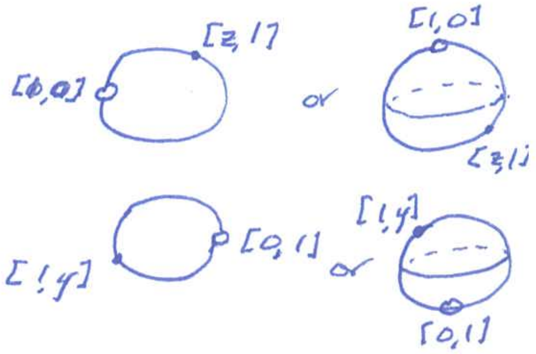
Then

$$G/B \simeq \mathbb{P}^1$$
$$gB \mapsto g[1,0] = [g_1, g_2] \quad \text{if } g = \begin{pmatrix} g_1 & * \\ g_2 & * \end{pmatrix} \in GL_2.$$

What are the points of $\mathbb{P}^1$?

$\mathbb{P}^1 = \{[1,0]\} \cup \{[z,1] \mid z \in \mathbb{C}\}$

$\mathbb{P}^1 = \{[1,y] \mid y \in \mathbb{C}\} \cup \{[0,1]\}$



A <u>chart</u> or <u>atlas</u> for $\mathbb{P}^1$ is

$\mathbb{P}^1 = U_1 \cup U_2$ (an open cover)

with $U_1 = \{[z,1] \mid z \in \mathbb{C}\}$
$U_2 = \{[1,y] \mid y \in \mathbb{C}\}$

and $U_1 \cap U_2 = \{[z,1] \mid z \in \mathbb{C}^\times\} = \{[1,y] \mid y \in \mathbb{C}^*\}$
with $[z,1] = [1, z^{-1}]$.

To see the G action we should use G/B

<u>Linear algebra Theorem 2</u> (LUP or LPU or UPU or LPL)

If

$$G = GL_n(\mathbb{C}) \text{ and } B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \text{ and } B^- = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}$$

then

$$G = \bigsqcup_{w \in S_n} BwB = \bigsqcup_{w \in S_n} B^- wB \qquad (*)$$

Let ususe $G = SL_2$, which is generated by

$$x_\alpha(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \text{ and } x_{-\alpha}(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

Let

$$h_{\alpha^\vee}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ and } n_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The elements $h_{\alpha^\vee}(t)$ and n_α are obtained from the generators $x_\alpha(t)$ and $x_{-\alpha}(t)$ by the identity

$$x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t) = h_{\alpha^\vee}(t)\, n_\alpha \quad \boxed{t \neq 0} \qquad (**)$$

for 2×2 matrices.

If $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2 \right\}$ then $(*)$ for this case is

$$G = B \cup Bn_\alpha B \qquad (1)$$

and

$$G = B^- B \cup n_\alpha B, \qquad (2)$$

where $B^- = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in SL_2 \right\}$.

More explicitly, $(**)$ says

$(-\mathrm{id})$

$$x_\alpha(t)\, n_\alpha = x_{-\alpha}(t^{-1})\, x_\alpha(-t)\, h_{\alpha^\vee}(-t) \quad \text{for } t \neq 0$$

so that $x_\alpha(z) n_\alpha B = x_{-\alpha}(z^{-1}) B$ $\left(\text{since } x_\alpha(-t) h_{\alpha^\vee}(-t) \in B\right)$.

Then

$$G = B \cup B n_\alpha B \quad \text{with } B n_\alpha B = \{ x_\alpha(z) n_\alpha B \mid z \in \mathbb{C} \}$$

and

$$G = B^- B \cup n_\alpha B \quad \text{with } B^- B = \{ x_{-\alpha}(y) B \mid y \in \mathbb{C} \}$$

So that G/B has a chart

$$G/B = B n_\alpha B \cup B^- B \quad \text{with}$$

$$U_1 = B n_\alpha B = \{ x_\alpha(z) n_\alpha B \mid z \in \mathbb{C} \}$$

$$U_2 = B^- B = \{ x_{-\alpha}(y) B \mid y \in \mathbb{C} \}$$

with $x_\alpha(z) n_\alpha B = x_{-\alpha}(z^{-1}) B$ if $z \neq 0$.

We have

$$G \supseteq B \supseteq T$$

with

$$G = SL_2(\mathbb{C}), \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, t \in \mathbb{C}^\times \right\}.$$

The irreducible rational representations of T are

$$X^k : T \longrightarrow \mathbb{C}^\times$$
$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto t^k, \quad \text{for } k \in \mathbb{Z}.$$

A tiny bit of K-theory

Let X be a space with a T-action.

$K_T(X)$ is the Grothendieck group of T-equivariant vector bundles on X.

Conceptually, $K_T(X)$ is a vector space with basis the simple T-equiv. vector bundles on X

and, in $K_T(X)$, if V is a T-equiv. vector bundle,

$$[V] = [S_1] + \cdots + [S_k] \quad \text{if } V \text{ is made of } S_1, \ldots, S_k$$

($S_1, \ldots, S_k$ are the "atoms" (simple vect. bundles) that make up the "molecule" V)

A vector bundle of rank n is a morphism of spaces with T-action $\begin{matrix} V \\ \downarrow p \\ X \end{matrix}$ such that the fibers $p^{-1}(x) \cong \mathbb{C}^n$ for $x \in X$.

SO: if X is a pt then $\begin{matrix} V \\ \downarrow \\ pt \end{matrix}$ is a vector space with a T-action,

i.e. a T-equivariant vector bundle on pt is a rational representation of T

and $K_T(pt) = \mathbb{C}[X^{\pm 1}] = \text{span}\{X^k \mid k \in \mathbb{Z}\}$

since the irreducible rational representations of T are the X^k, $k \in \mathbb{Z}$.

The line bundles $\mathcal{L}_k$ on G/B

The representation

$$\begin{aligned} X^k : T &\longrightarrow \mathbb{C}^\times \\ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &\longmapsto t^k \end{aligned}$$

corresponds to the 1-dimensional T-module

$$\mathbb{C}_k = \mathrm{span}\{v_k\} \quad \text{with} \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} v_k = t^k v_k$$

$\mathbb{C}_k$ extends to a B-module by

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} v_k = t^k v_k, \qquad \text{i.e.} \quad \begin{aligned} X^k : B &\longrightarrow \mathbb{C}^\times \\ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} &\longmapsto t^k. \end{aligned}$$

Then

$$K_T(pt) = K_B(pt)$$

and

$$\begin{aligned} K_T(pt) = K_B(pt) &\longrightarrow K_G(G/B) \\ \mathbb{C}_k &\longmapsto G \times_B \mathbb{C}_k \end{aligned}$$

where

$$\mathcal{L}_k = G \times_B \mathbb{C}_k = \frac{G \times \mathbb{C}_k}{\langle (gb, cv_k) = (g, bcv_k) \rangle}$$

and

$$\begin{array}{ccc} G \times_B \mathbb{C}_k & & (g, cv_k) \\ \pi \downarrow & & \downarrow \\ G/B & & gB. \end{array}$$

The $\mathcal{L}_k$ are line bundles (vector bundles of rank 1) on G/B (i.e. $\mathbb{P}^1$).

Sections of $\mathcal{L}_k$

Recall

$$G/B = B \cup Bn_\alpha B \quad \text{with } Bn_\alpha B = \{x_\alpha(z)n_\alpha B \mid z \in \mathbb{C}\}$$

$$= B^- B \cup n_\alpha B \quad \text{with } B^- B = \{x_{-\alpha}(y)B \mid y \in \mathbb{C}\}$$

and $x_\alpha(z)n_\alpha B = x_{-\alpha}(z^{-1})B$ if $z \neq 0$.

Then

$$\mathcal{L}_k = G \times_B \mathbb{C}_k = \{(1, c'v_k) \mid c' \in \mathbb{C}\} \cup \{(x_\alpha(z)n_\alpha, cv_k) \mid z, c \in \mathbb{C}\}$$

$$= \{(x_{-\alpha}(y), c'v_k) \mid y, c' \in \mathbb{C}\} \cup \{(n_\alpha, cv_k) \mid c \in \mathbb{C}\}$$

with

$$(x_\alpha(z)n_\alpha, cv_k) = (x_{-\alpha}(z^{-1})x_\alpha(z)h_{\alpha^\vee}(-z), cv_k)$$

$$= (x_{-\alpha}(z^{-1}), c(-z)^k v_k), \quad \text{if } z \neq 0.$$

(this is the transition/clutching identity for $\mathcal{L}_k$).

A <u>global section</u> of $\mathcal{L}_k$ is a map $s: G/B \to \mathcal{L}_k$ such that $\pi \circ s = id_{G/B}$

$$\begin{array}{ccc} G \times_B \mathbb{C}_k & & (g, f(g)v_k) \\ \pi \big\downarrow \big\uparrow s & & \updownarrow \\ G/B & & gB \end{array}$$

In order for $f: G \to \mathbb{C}$ to correspond to a section s we need

$$(gb, f(gb)v_k) = (g, f(g)v_k)$$

i.e.

$$(g, f(g)v_k) = (gb, f(gb)v_k) = (g, f(gb)\, b\, v_k)$$
$$= (g, f(gb)\, X^k(b)\, v_k).$$

So

$$\left\{\begin{matrix}\text{global sections}\\ s \text{ of } \mathcal{L}_k\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{functions } f: G \to \mathbb{C}\\ \text{such that}\\ f(g) = f(gb)X^k(b), \text{ for } b \in B\end{matrix}\right\}$$

Use the notation

$$H^0(G/B, \mathcal{L}_k) = \left\{\begin{matrix}\text{global sections}\\ s \text{ of } \mathcal{L}_k\end{matrix}\right\}.$$

Any function $f: G \to \mathbb{C}$ that satisfies

$$f(g) = f(gb)X^k(b) \quad \text{for } b \in B \qquad (C)$$

is determined by its values on coset representatives of cosets in G/B. So f is determined by

$f_1(z) = f(x_\alpha(z)n_\alpha)$, $\quad f_1: \mathbb{C} \to \mathbb{C}$ a rational function defined at $z=0$

or by

$f_2(y) = f(x_\alpha(y))$, $\quad f_2: \mathbb{C} \to \mathbb{C}$ a rational function defined at $y=0$

So f_1 is a polynomial in z

and f_2 is a polynomial in y

and

$$f_1(z) = f(x_\alpha(z)n_\alpha) = f(x_{-\alpha}(z^{-1})x_\alpha(z)h_{\alpha^\vee}(-z))$$
$$= f(x_{-\alpha}(z^{-1}))X^k(x_\alpha(z)h_{\alpha^\vee}(-z)) = f(x_{-\alpha}(z^{-1}))(-z)^k$$
$$= f_2(z^{-1})(-z^k).$$

So elements of $H^0(G/B, \mathcal{L}_k)$ correspond to

$$f_2 \in \mathbb{C}[y] \text{ such that } f_2(z^{-1})(-z)^k \in \mathbb{C}[z].$$

So

$$f_2 \in \operatorname{span}\{1, y, y^2, \ldots, y^k\}$$

and

$$\dim(H^0(G/B, \mathcal{L}_k)) = k+1, \quad \text{if } k \in \mathbb{Z}_{\geq 0}$$

$$\text{and } \dim(H^0(G/B, \mathcal{L}_k)) = 0, \quad \text{if } k \in \mathbb{Z}_{<0}.$$

Let $b_0, b_1, \ldots, b_k$ be the functions $b_i : G \to \mathbb{C}$ given by

$$b_i(x_{-\alpha}(y)) = y^i \text{ and condition (C).}$$

The group G acts on $H^0(G/B, \mathcal{L}_k)$ by

$$(hf)(g) = f(h^{-1}g), \quad \text{for } f : G \to \mathbb{C} \text{ and } h \in G.$$

Then

$$\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} b_i\right)(x_{-\alpha}(y)) = b_i\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} x_{-\alpha}(y)\right)$$

$$= b_i\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} t^{+1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)$$

$$= b_i\left(x_{-\alpha}(t^2 y)\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = (t^2 y)^i t^{-k}$$

$$= t^{2i-k} y^i = t^{2i-k} b_i(x_{-\alpha}(y)).$$

So $\mathbb{C}b_i$ is one dimensional T-module with character X^{2i-k}

So $H^0(G/B, \mathcal{L}_k) = X^{-k} + X^{-(k-2)} + \cdots + X^{k-2} + X^k$ in $K_T(pt)$.

Then

$$\left(\left(\begin{smallmatrix}1&0\\w&1\end{smallmatrix}\right)b_i\right)(x_{-\alpha}(y)) = b_i\left(\left(\begin{smallmatrix}1&0\\-w&1\end{smallmatrix}\right)x_{-\alpha}(y)\right) = b_i(x_{-\alpha}(y-w))$$

$$= (y-w)^i = y^i - \binom{i}{1}y^{i-1}w + \cdots + \binom{i}{i}(-w)^i$$

$$= \left(b_i - \binom{i}{1}w\,b_{i-1} + \binom{i}{2}w^2 b_{i-2} + \cdots + (-w^i)b_0\right)(x_{-\alpha}(y)).$$

So b_0 is the unique vector invariant under the action of $\mathcal{U}^- = \left\{\left(\begin{smallmatrix}1&0\\y&1\end{smallmatrix}\right) \mid y\in\mathbb{C}\right\}$

Similary, b_k is the unique vector invariant under the action of $\mathcal{U}^+ = \left\{\left(\begin{smallmatrix}1&z\\0&1\end{smallmatrix}\right) \mid z\in\mathbb{C}\right\}$.

So $H^0(G/B, \mathcal{L}_k)$ is an SL_2-module with a unique highest weight vector, of weight χ^k.