CBMS Lecture 2: ~~The Weyl Character formula~~
~~The Weyl character formula from the affine Hecke algebra point of view~~ (1)
Symmetric functions
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The initial data is $(W_0, \mathfrak{h}_{\mathbb{Z}})$, a finite $\mathbb{Z}$-reflection group, i.e.

$\mathfrak{h}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module,

W_0 a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by reflections.

<u>Example</u> $\mathfrak{h}_{\mathbb{Z}} = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ with

$W_0 = S_n$ acting by permuting $\varepsilon_1, \ldots \varepsilon_n$.

The group algebra of $\mathfrak{h}_{\mathbb{Z}}$ is

$$\mathbb{C}[X] = \mathrm{span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}\} \text{ with } X^\lambda X^\mu = X^{\lambda+\mu}$$

W_0 acts on $\mathbb{C}[X]$ by $wX^\lambda = X^{w\lambda}$.

The <u>ring of symmetric functions</u> is

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] \mid wf = f\}$$

<u>Example</u> <u>Type GL_3</u>: Let $z_1 = X^{\varepsilon_1}$, $z_2 = X^{\varepsilon_2}$, $z_3 = X^{\varepsilon_3}$

Then $W_0 = S_3$, $\mathbb{C}[X] = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$ and

$$\mathbb{C}[X]^{W_0} = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]^{S_3} = \mathbb{C}[e_1, e_2, e_3^{\pm 1}],$$

where

$$e_1 = z_1 + z_2 + z_3, \quad e_2 = z_1 z_2 + z_1 z_3 + z_2 z_3, \quad e_3 = z_1 z_2 z_3.$$

Weyl characters

Let C_0 be a fundamental region for W_0 acting on $\mathfrak{h}_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z}$. Let

$$\mathfrak{h}_\mathbb{Z}^+ = \mathfrak{h}_\mathbb{Z} \cap \overline{C_0} \quad \text{and} \quad \mathfrak{h}_\mathbb{Z}^{++} = \mathfrak{h}_\mathbb{Z} \cap C_0$$

where $\overline{C_0}$ is the closure of C_0. Then

$$\begin{array}{ccc} \mathfrak{h}_\mathbb{Z}^+ & \xrightarrow{\sim} & \mathfrak{h}_\mathbb{Z}^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{as } \mathfrak{h}_\mathbb{Z}^+\text{-modules}$$

Define

$$\mathbb{C}[X]^{\det} = \{ f \in \mathbb{C}[X] \mid wf = \det(w) f, \text{ for } w \in W_0 \}$$

<u>Theorem</u> $\mathbb{C}[X]^{\det}$ is a free $\mathbb{C}[X]^{W_0}$ module of rank 1.

$$\begin{array}{ccc} \mathbb{C}[X]^{W_0} & \xrightarrow{\sim} & \mathbb{C}[X]^{\det} \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longleftarrow\!\!\shortmid & a_{\lambda+\rho} \end{array} \qquad \text{as } \mathbb{C}[X]^{W_0}\text{-modules}$$

"naive basis"

"naive basis" $\quad m_\lambda$

where $m_\lambda = \sum_{\gamma \in W_0 \lambda} X^\gamma$ and $a_\mu = \sum_{w \in W_0} \det(w) X^{w\mu}$.

The <u>Weyl character</u> is

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in \mathfrak{h}_\mathbb{Z}^+$$

The affine Hecke algebra H.

Let $\mathfrak{h}^{\alpha_1}, \dots, \mathfrak{h}^{\alpha_\ell}$ be the walls of C_0

$s_1, \dots, s_\ell$ the corresponding reflections, so that

$s_i : \mathfrak{h}_{\mathbb{Z}} \to \mathfrak{h}_{\mathbb{Z}}$ is given by

$$s_i \lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \text{for } i = 1, \dots, \ell.$$

The affine Hecke algebra H is generated by

$T_1, \dots, T_\ell$ and X^λ, $\lambda \in \mathfrak{h}_{\mathbb{Z}}$

with relations

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1, \quad \text{for } i = 1, \dots, \ell$$

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}, \quad \text{for } i \neq j \text{ with } \frac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}.$$

$$X^\lambda X^\mu = X^{\lambda + \mu}, \quad \text{for } \lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}$$

$$T_i X^\lambda = X^{s_i \lambda} T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{X^\lambda - X^{s_i \lambda}}{1 - X^{-\alpha_i}}$$

Define

$T_w = T_{i_1} \cdots T_{i_\ell}$ for a reduced word $w = s_{i_1} \cdots s_{i_\ell}$.

Then $\{ X^\lambda T_w \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0 \}$ is a basis of H

and

$\mathbb{C}[X] = \operatorname{span}\{ X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}} \}$ and $H_0 = \operatorname{span}\{ T_w \mid w \in W_0 \}$

are subalgebras.

Bernstein-Satake-Lusztig isomorphisms

Let $\mathbb{1}_0, \varepsilon_0 \in H_0$ be such that

$$\begin{matrix} \mathbb{1}_0^2 = \mathbb{1}_0 \\ \varepsilon_0^2 = \varepsilon_0 \end{matrix} \quad \text{and} \quad \begin{matrix} T_i \mathbb{1}_0 = t^{\frac{1}{2}} \mathbb{1}_0 \\ T_i \varepsilon_0 = (-t^{-\frac{1}{2}}) \varepsilon_0 \end{matrix} \quad \text{for } i = 1, \dots, l.$$

Then

$$\begin{matrix} H & \longrightarrow & H\mathbb{1}_0 = \mathbb{C}[X]\mathbb{1}_0 \\ h & \longmapsto & h\mathbb{1}_0 \end{matrix}$$

makes $\mathbb{C}[X]$ into an H-module (the <u>polynomial representation</u>). Then

$$\begin{array}{ccccccc}
\mathbb{C}[X]^{W_0} & \overset{\text{Bernstein}}{=\!=} & Z(H) & \xrightarrow[\sim]{\text{Satake}} & \mathbb{1}_0 H \mathbb{1}_0 & \xrightarrow[\sim]{\text{Lusztig}} & \varepsilon_0 H \mathbb{1}_0 \\
f & \longmapsto & f & \longmapsto & f\mathbb{1}_0 & \longmapsto & A_\rho f \mathbb{1}_0 \\
 & & & & & & \\
s_\lambda & \longleftarrow\! & & \!\longleftarrow\!\!\!\shortmid & C_\lambda & \longleftarrow\!\!\!\shortmid & A_{\lambda+\rho} \\
P_\lambda(0,t) & \longleftarrow\! & & \!\longleftarrow\!\!\!\shortmid & M_\lambda & &
\end{array}
\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \text{"naive bases".}$$

where $M_\lambda = \mathbb{1}_0 X^\lambda \mathbb{1}_0$ and $A_\mu = \varepsilon_0 X^\mu \mathbb{1}_0$

C_λ is the <u>Kazhdan-Lusztig basis</u> of the <u>spherical Hecke</u> algebra $\mathbb{1}_0 H \mathbb{1}_0 = K_0(\mathrm{Perv}(G/K))$ the Grothendieck group of the category $\mathrm{Perv}(G/K)$ of perverse sheaves on the <u>loop Grassmannian</u> G/K

$P_\lambda(0,t)$ is <u>Macdonald's spherical function</u> for $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$

Weyl's Theorems

Let $G(\mathbb{C})$ be the reductive algebraic group corresponding to $(W_0, \mathfrak{h}_{\mathbb{Z}})$.

(a) The simple T-modules X^λ are indexed by $\mathfrak{h}_{\mathbb{Z}}^*$

(b) The simple G-modules $L(\lambda)$ are indexed by $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$.

(c) The character of $L(\lambda)$ is

$$\mathrm{Res}^G_T(L(\lambda)) = s_\lambda$$

(d)
$$a_\rho = X^\rho \prod_{\alpha \in R^+} (1 - X^{-\alpha})$$

where R^+ is an index set for the reflections s_α in W_0, so that

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$

q-Weyl denominator

$$A_\rho = \prod_{\alpha \in R^+} \left(t^{\frac{1}{2}} X^{\alpha/2} - t^{-\frac{1}{2}} X^{-\alpha/2}\right)$$