$\mathbb{Z}_{\geq 0}$ is the free monoid (semigroup) generated by 1.

$\mathbb{C}[t] = \text{span}\{t^i \mid i \in \mathbb{Z}_{\geq 0}\}$ with $t^i t^j = t^{i+j}$

is the group algebra of $\mathbb{Z}_{\geq 0}$.

By definition,

$$\{\mathbb{Z}_{\geq 0}\text{-modules}\} = \{\mathbb{C}[t]\text{-modules}\}.$$

A $\underline{\mathbb{C}[t]\text{-module}}$ is a vector space V over $\mathbb{C}$ with an action of $\mathbb{C}[t]$,

$$\begin{matrix} \mathbb{C}[t] \otimes V \longrightarrow V \\ p \otimes v \longmapsto pv \end{matrix} \qquad \text{such that}$$

(a) If $p_1, p_2 \in \mathbb{C}[t]$ then $p_1(p_2 v) = (p_1 p_2) v$,

(b) If $v \in V$ then $1v = v$

(c) The action is linear.

A $\mathbb{C}[t]$-action on V is determined by the action of t on V, which is a linear transformation $\varphi: V \longrightarrow V$. So

$$\{\mathbb{C}[t]\text{-modules}\} \longleftrightarrow \left\{\begin{matrix}\text{Pairs } (V,\varphi) \text{ with} \\ V \text{ a vector space} \\ \varphi \text{ a linear transformation} \\ \varphi: V \longrightarrow V\end{matrix}\right\}$$

$\mathbb{C}[t] \otimes V \to V$
given by
$(3t^2 + t^3 + 7)v = (3\varphi^2 + \varphi^3 + 7)v \longleftarrow\!\shortmid (V, \varphi)$

$$\left\{\begin{array}{l}\text{Triples } (V,\varphi,B) \text{ with} \\ V \text{ a vector space} \\ \varphi: V\to V \text{ a lin. transf} \\ B \text{ a basis of } V\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{Pairs } (n,a) \text{ with} \\ n\in\mathbb{Z}_{\geq 0} \text{ and } a\in M_n(\mathbb{C})\end{array}\right\}$$

Define a $GL_n(\mathbb{C})$ action on $M_n(\mathbb{C})$ by

$$Ad_g(a) = gag^{-1}, \qquad \text{for } g\in GL_n(\mathbb{C}),\ a\in M_n(\mathbb{C})$$

and let

$$\begin{aligned}[a] &= \{gag^{-1} \mid g\in GL_n(\mathbb{C})\} = Ad_G(a) \\ &= G\text{-orbit of } a \\ &= \text{equivalence class of } a \text{ under conjugation.}\end{aligned}$$

Then

$$\{\mathbb{C}[t]\text{-modules}\} \longleftrightarrow \left\{\begin{array}{l}\text{Pairs } (V,\varphi): \\ V \text{ a vector space} \\ \varphi: V\to V \text{ a linear transf.}\end{array}\right\}$$

$$\longleftrightarrow \left\{\begin{array}{l}\text{Pairs } (n,a) \text{ with} \\ n\in\mathbb{Z}_{\geq 0} \text{ and } a\in M_n(\mathbb{C})\end{array}\right\} \Big/ \text{conjugation}$$

Let V be a $\mathbb{C}[t]$-module. The <u>annihilator</u> of V is

$$\mathrm{ann}(V) = \{p\in\mathbb{C}[t] \mid \text{if } v\in V \text{ then } pv = 0\}.$$

<u>Lemma</u> $\mathrm{ann}(V)$ is an ideal in $\mathbb{C}[t]$.

<u>Proof:</u> To show (a) If $p_1, p_2 \in \mathrm{ann}(V)$ then $p_1 + p_2 \in \mathrm{ann}(V)$

(b) If $f \in \mathbb{C}[t]$ and $p \in \mathrm{ann}(V)$ then $fp \in \mathrm{ann}(V)$ //

<u>Lemma</u> $\mathbb{C}[t]$ is a PID. (every ideal is generated by one element).

Let V be a $\mathbb{C}[t]$-module. Let φ be the ~~irreducible polynomial~~ linear transformation given by the action of t on V. The <u>minimal polynomial of φ</u> is

$m \in \mathbb{C}[t]$ such that $\mathrm{ann}(V) = m\mathbb{C}[t] = (m)$.

Let A be an algebra

A <u>simple</u> A-module is an A-module M

that has no submodules except $\{0\}$ and M.

What are the simple $\mathbb{C}[t]$-modules?

(1) $\mathbb{C}_a = \mathrm{span}\{v_a\}$ with $tv_a = av_a$, for $a \in \mathbb{C}$.

are simple modules.

(2) If V is a $\mathbb{C}[t]$-module with

$\mathrm{ann}(V) = (m)$ where $m = (t-a_1)(t-a_2)\cdots(t-a_\ell)$

then there exists $v \in V$ such that

$(t-a_2)(t-a_3)\cdots(t-a_\ell)\cdot v \neq 0$ and $(t-a_1)((t-a_2)\cdots(t-a_\ell)v) = 0$.

So $t\cdot((t-a_2)\cdots(t-a_\ell)v) = a_1((t-a_2)\cdots(t-a_\ell)v)$

So V contains a simple submodule (an eigenvector).

In other words:

$$\left\{\begin{array}{l}\text{eigenvectors of} \\ \varphi \text{ in } V\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{simple submodules} \\ \text{of the } \mathbb{C}[t]\text{-module } V\end{array}\right\}$$

Let $\lambda \in \mathbb{C}$. Let V be a $\mathbb{C}[t]$-module

The λ-weightspace, or λ-eigenspace, of V is

$$V_\lambda = \{v \in V \mid tv = \lambda v\}.$$

The λ-generalised eigenspace, or λ-generalised weight space of V is

$$V_\lambda^{gen} = \{v \in V \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ with } (t-\lambda)^k v = 0\}$$

Let

$$V_\lambda^k = \{v \in V \mid (t-\lambda)^k v = 0\}$$

Then $V_\lambda \subseteq V_\lambda^2 \subseteq V_\lambda^3 \subseteq \cdots \subseteq V_\lambda^{gen}$ as $\mathbb{C}[t]$-modules.

Theorem $\displaystyle V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^{gen}$ as $\mathbb{C}[t]$-modules.

Proof Let m be the minimal polynomial of φ.
By partial fractions: If $m = (x-\lambda_1)^{\nu_1} \cdots (x-\lambda_r)^{\nu_r}$ then

$$\frac{1}{m} = \frac{A_1}{(x-\lambda_1)^{\nu_1}} + \cdots + \frac{A_r}{(x-\lambda_r)^{\nu_r}}$$

where
$$A_j = \sum_{k=0}^{\nu_j - 1} \frac{1}{k!} \left. \frac{d^k \left(\frac{(x-\lambda_j)^{\nu_j}}{m} \right)}{dx^k} \right|_{x=\lambda_i} (x-\lambda_i)^k .$$

Then
$$1 = A_1 \left(\frac{m}{(x-\lambda_1)^{\nu_1}} \right) + \cdots + A_r \frac{m}{(x-\lambda_r)^{\nu_r}}$$

~~Then~~ as operators on V and $\left(A_i \frac{m}{(x-\lambda_i)^{\nu_i}} \right) \left(A_j \frac{m}{(x-\lambda_j)^{\nu_j}} \right) = 0$

and $\left(A_i \frac{m}{(x-\lambda_i)^{\nu_i}} \right)^2 = A_i \frac{m}{(x-\lambda_i)^{\nu_i}}$

as operators on V. and

$$\left(A_i \frac{m}{(x-\lambda_i)^{\nu_i}} \right) \cdot V = V_{\lambda_i}^{gen} .$$

So $V = V_{\lambda_1}^{gen} \oplus V_{\lambda_2}^{gen} \oplus \cdots \oplus V_{\lambda_r}^{gen}$. //.

What is the <u>effective</u> partial fractions for $\mathbb{Z}$?