Lecture 11.08.2011 Representation theory.

The degenerate affine Hecke algebra $\mathcal{H}_d$ has generators

$$s_1, \ldots, s_{d-1} \quad \text{and} \quad x_1, \ldots, x_d$$

with relations

$$x_r x_s = x_s x_r, \quad s_r s_{r+1} s_r = s_{r+1} s_r s_{r+1}, \quad s_r s_t = s_t s_r \text{ if } t \neq r \pm 1$$

$$s_r^2 = 1, \quad \cancel{x_s x_r = x} \; s_r x_s = x_s s_r \text{ if } s \neq r, r+1$$

$$s_r x_{r+1} = x_r s_r + 1.$$

Let $\Lambda = \Lambda_0$ with $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$.

Let $H_d^{\Lambda_0} = \dfrac{H_d}{\langle \prod_{i \in I} (x_1 - i)^{\langle \Lambda_0, \alpha_i^\vee \rangle} \rangle} = \dfrac{H_d}{\langle x_1 - 0 \rangle} = \mathbb{C} S_d.$

Define

$$e_{\underline{i}} = \prod (x_r - i_r)\,?$$

Not really needed. $\left[\; \varphi_r = s_r + \sum_{i_r \neq i_{r+1}} \frac{1}{x_r - x_{r+1}} e_{\underline{i}} + \sum_{i_r = i_{r+1}} e_{\underline{i}} = \sum_{\underline{i}} (s_r + p_r(\underline{i})) e_{\underline{i}} \right.$

$$y_r = \sum_{\underline{i}} (x_r - i_r) e_{\underline{i}}$$

$$\psi_r = \sum_{\underline{i}} (s_r + p_r(\underline{i})) \frac{1}{q_r(\underline{i})} e_{\underline{i}}$$

Theorem (Brundan-Kleshchev) The group algebra of S_n is presented by generators

$$y_1, \dots, y_n, \quad \psi_1, \dots, \psi_{n-1} \quad \text{and} \quad e(i_1, \dots, i_n), \text{ where } i_1, \dots, i_n \in \mathbb{Z}.$$

with relations

$$e_{\underline{i}} e_{\underline{j}} = \delta_{\underline{i}\underline{j}} e_{\underline{i}}, \quad 1 = \sum_{\underline{i}} e_{\underline{i}}, \quad y_r e_{\underline{i}} = e_{\underline{i}} y_r$$

$$e_{\underline{i}} \psi_r = \psi_r e_{s_r \underline{i}}, \quad y_r y_s = y_s y_r, \quad \psi_r y_s = y_s \psi_r \text{ if } s \neq r, r+1$$

$$\psi_r y_r e_{\underline{i}} = \begin{cases} (y_{r+1} \psi_r + 1) e_{\underline{i}}, & \text{if } i_{r+1} = i_r + 1 \\ (y_{r+1} \psi_r - 1) e_{\underline{i}}, & \text{if } i_{r+1} = i_r - 1 \\ y_{r+1} \psi_r e_{\underline{i}}, & \text{otherwise} \end{cases}$$

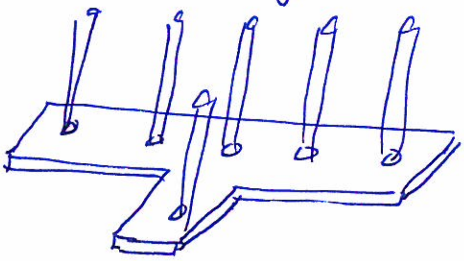
$$\psi_r^2 e_{\underline{i}} = \begin{cases} \pm (y_r - y_{r+1}) e_{\underline{i}}, & \text{if } i_{r+1} = i_r \pm 1, \\ 0, & \text{if } i_{r+1} = i_r \\ e_{\underline{i}}, & \text{otherwise} \end{cases}$$

$$\psi_r \psi_{r+1} \psi_r e_{\underline{i}} = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} \pm 1) e_{\underline{i}}, & \text{if } (i_r, i_{r+1}, i_{r+2}) = (i_r, i_r \pm 1, i_r), \\ \psi_{r+1} \psi_r \psi_{r+1} e_{\underline{i}}, & \text{otherwise.} \end{cases}$$

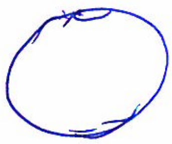
and

$$y_1 e_{\underline{i}} = 0 \text{ if } i_1 = 0, \quad \text{and} \quad e_{\underline{i}} = 0 \text{ if } i_1 \neq 0.$$

The glass bead game



Board

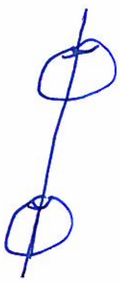
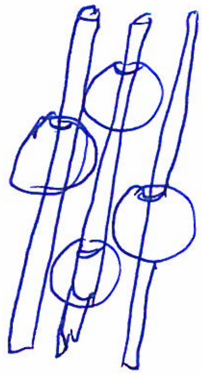
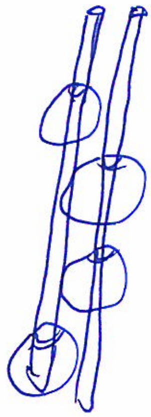


Beads.

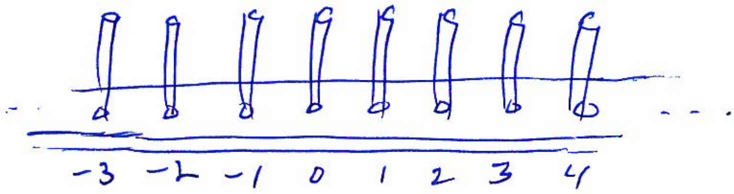
A *shape* λ is a configuration of beads

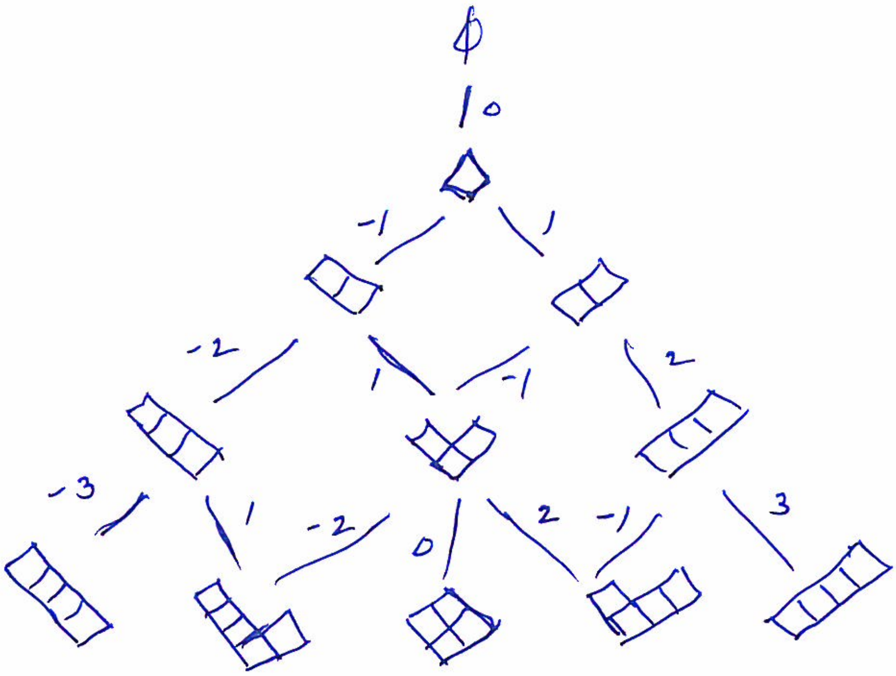
A *standard* *tableau* of shape λ is a sequence $i_1, \dots, i_k$ of runners such that playing the sequence $i_1, \dots, i_k$ results in the shape λ.

A *skew* *shape* is λ such that any two beads on the same runner are separated by at least two beads

if  then  or 

The Young lattice





Define a vector space

$$S_n^\lambda = \mathrm{span}\left\{ v_{i_1\cdots i_n} \;\middle|\; \begin{array}{l}(i_1,\dots,i_n) \text{ is a standard} \\ \text{tableau of shape } \lambda\end{array} \right\}$$

Standard tableaux of shape λ correspond to paths from $\emptyset$ to λ in the Young lattice.

Define an action of $\mathbb{C}S_n$ on S_n^λ by

$$e_j v_{\underline{i}} = \delta_{\underline{i}j} v_{\underline{i}}, \qquad y_r v_{\underline{i}} = 0$$

$$\psi_r v_{\underline{i}} = \begin{cases} v_{s_r\underline{i}}, & \text{if } s_r\underline{i} \text{ is a standard tableau of shape } \lambda \\ 0, & \text{otherwise} \end{cases}$$

$\deg(v_{\underline{i}}) = 0.$

<u>Theorem</u> (Young, Kleshchev-Ram)

The S_n^λ are the simple S_n-modules.