Representation Theory: Tuesday Branching rules

There are equivalences of categories:

$$\left\{\begin{matrix}\text{complex reductive}\\ \text{algebraic groups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{compact}\\ \text{Lie groups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-reflection}\\ \text{groups}\end{matrix}\right\}$$

$$GL_n(\mathbb{C}) \longmapsto U_n(\mathbb{C}) \longmapsto (S_n, \mathbb{Z}^n)$$

$G \longmapsto K$, the maximal compact subgroup $\longmapsto (W_0, \mathring{\mathfrak{h}}^*_{\mathbb{Z}})$ with

$W_0 = N(T)/T$, $\mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^\times)$.

Hence there are equivalences:

$$\{G\text{-modules}\} \longleftrightarrow \{K\text{-modules}\} \longleftrightarrow \{(W_0, \mathring{\mathfrak{h}}_{\mathbb{Z}})\text{ crystals}\}$$

$$L(\lambda) \longmapsto L(\lambda) \longmapsto B(\lambda)$$

Suppose $H \subseteq G$ (or $L \subseteq K$) (or $(W_J, \mathring{\mathfrak{h}}_{\mathbb{Z}}) \subseteq (W_0, \mathring{\mathfrak{h}}_{\mathbb{Z}})$)

Let $L(\lambda)$ be a simple G-module:

Describe the decomposition of

$$\mathrm{Res}^G_H(L(\lambda)) \text{ into irreducibles.}$$

This should be equivalent to

$$\mathrm{Res}^{(W_0, \mathring{\mathfrak{h}}_{\mathbb{Z}})}_{(W_J, \mathring{\mathfrak{h}}_{\mathbb{Z}})}(B(\lambda)).$$

Example The adjoint representation of $SU_5(\mathbb{C})$ restricted to $SU_3 \times SU_2 \times U_1$