Questions for Assignment 4

MAST90017 Representation Theory
Semester II 2015
Lecturer: Arun Ram
to be turned in on 27 August 2015 before 5pm

(1) (The GL_n-crystal $B^{\otimes k}$) Let p_i be the straight line path to ε_i and let $B = \{p_1, \ldots, p_n\}$ be the GL_n-crystal generated by p_1. Show that the crystal action on the tensor product $B^{\otimes k}$ is given by the following:

For $i \in \{1, \ldots, n-1\}$ define

$$\tilde{f}_i \colon B(p_1)^{\otimes k} \longrightarrow B(p_1)^{\otimes k} \cup \{0\} \quad \text{and} \quad \tilde{e}_i \colon B(p_1)^{\otimes k} \longrightarrow B(p_1)^{\otimes k} \cup \{0\}$$

as follows. For $b \in B(p_1)^{\otimes k}$,

place $+1$ under each p_i in b,
place -1 under each p_{i+1} in b, and
place 0 under each p_j, $j \neq i, i+1$.

Ignoring 0s, successively pair adjacent $(+1, -1)$ pairs to obtain a sequence of unpaired -1s and $+1$s

$$-1 \ -1 \ -1 \ -1 \ -1 \ -1 \ -1 \ +1 \ +1 \ +1 \ +1$$

(after pairing and ignoring 0s). Then

$\tilde{f}_i b =$ same as b except the letter corresponding to the leftmost unpaired $+1$ is changed to p_{i+1},
$\tilde{e}_i b =$ same as b except the letter corresponding to the rightmost unpaired -1 is changed to p_i.

If there is no unpaired $+1$ after pairing then $\tilde{f}_i b = 0$.
If there is no unpaired -1 after pairing then $\tilde{e}_i b = 0$.

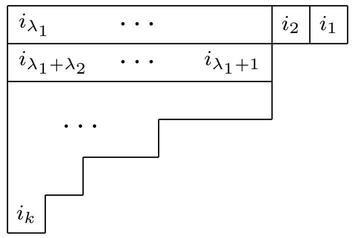
Before launching the general proof do some illustrative small and smallish examples.

(2) (The GL_n-crystal $B(\lambda)$) Let λ be a partition with k boxes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$

The set $B(\lambda)$ is a subset of $B(\varepsilon_1)^{\otimes k}$ via the injection

$$\begin{array}{ccc} B(\lambda) & \hookrightarrow & B(\varepsilon_1)^{\otimes k} \\ p & \longmapsto & (\text{the arabic reading of } p) \end{array}$$



$$\longmapsto \quad \varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_k} \tag{0.1}$$

where the arabic reading of p is $\varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_k}$ if the entries of p are $i_1, i_2, \ldots, i_k$ read right to left by rows with the rows read in sequence beginning with the first row. Show that this embedding determines a GL_n-crystal structure on $B(\lambda)$, in particular that $B(\lambda)$ is closed under the action of $\tilde{e}_1, \ldots, \tilde{e}_{n-1}, \tilde{f}_1, \ldots, \tilde{f}_{n-1}$. Before launching the general proof do some illustrative small and smallish examples.

(3) (RSK insertion) Let λ be a partition with k boxes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$

Give (with proof) a crystal isomorphism

$$B(\lambda) \otimes B \cong \bigoplus_{\mu/\lambda=\square} B(\mu),$$

where the sum is over partitions μ obtained from λ by adding a box. Before launching the general proof do some illustrative small and smallish examples.

(4) (RSK insertion) Let λ be a partition with k boxes and let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda\}.$$

Give (with proof) a crystal isomorphism

$$B \otimes B(\lambda) \cong \bigoplus_{\mu/\lambda=\square} B(\mu),$$

where the sum is over partitions μ obtained from λ by adding a box. Before launching the general proof do some illustrative small and smallish examples.

(5) (Knuth transformations) Define S_k-crystal operators on $B^{\otimes k}$ so that the RSK bijection

$$B^{\otimes k} \cong \bigoplus_{\lambda \vdash k} B(\lambda) \otimes S(\lambda).$$

is a $(GL_n \times S_k)$-crystal isomorphism. Before launching the general proof do some illustrative small and smallish examples.

(6) (The dimension of $\mathbb{C}S_k$) Let f^λ be the number of standard tableaux of shape λ. Prove that

$$k! = \sum_{\lambda \vdash k} (f^\lambda)^2.$$

Before launching the general proof do some illustrative small and smallish examples.

(7) (The character of $B^{\otimes k}$) Let f^λ be the number of standard tableaux of shape λ. Prove that, as elements of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{\lambda \vdash k} f^\lambda s_\lambda.$$

Before launching the general proof do some illustrative small and smallish examples.

(8) (The dimension of $B^{\otimes k}$) Let f^λ be the number of standard tableaux of shape λ. Let d_λ be the number of column strict tableaux of shape λ filled from $\{1, \ldots, n\}$. Prove that

$$n^k = \sum_{\lambda \vdash k} f^\lambda d_\lambda.$$

Before launching the general proof do some illustrative small and smallish examples.

(9) (The Weyl denominator) Let

$$\begin{aligned} \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\det} &= \{ f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \text{if } w \in S_n \text{ then } wf = \det(w) f\}, \text{ and} \\ \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} &= \{ f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \text{if } w \in S_n \text{ then } wf = f\}, \end{aligned}$$

and let

$$a_\rho = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j).$$

Show that the map

$$\begin{array}{ccc} \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W & \longrightarrow & \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\det} \\ f & \longmapsto & a_\rho f \end{array}$$

is an isomorphism of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ modules ($\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ acts by left multiplication). Before launching the general proof do some illustrative small and smallish examples.

(10) (The Vandermonde determinant) Show that, as elements of $\mathbb{C}[x_1, \ldots, x_n]$,

$$\det(x_i^{n-j}) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j).$$

Before launching the general proof do some illustrative small and smallish examples.

(11) (The Weyl character formula) Prove that, as elements of $\mathbb{C}[x_1, \ldots, x_n]$,

$$s_\lambda = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})},$$

where, as defined in class s_λ is the character of the GL_n-crystal $B(\lambda)$. Before launching the general proof do some illustrative small and smallish examples.

(12) (The Cauchy identity) Show that, as elements of $\mathbb{C}[[x_1, \ldots, x_m, y_1, \ldots, y_m]]$,

$$\prod_{i=1}^{m}\prod_{j=1}^{n} \frac{1}{1-x_iy_j} = \sum_\lambda s_\lambda(x)s_\lambda(y),$$

where $s_\lambda(x) \in \mathbb{C}[x_1, \ldots, x_m]$ is the Schur function labeled by λ in $x_1, \ldots, x_m$ and $s_\lambda(y)$ is the Schur function labeled by λ in $y_1, \ldots, y_n$ and the sum is over partitions λ with at most $\min(m, n)$ rows.

(13) (The dual Cauchy identity) For a partition λ, let λ' be the *conjugate*, or *transpose*, partition, obtained by flipping λ about the main diagonal. Show that, as elements of $\mathbb{C}[[x_1, \ldots, x_m, y_1, \ldots, y_m]]$,

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda'}(y),$$

where $s_\lambda(x) \in \mathbb{C}[x_1, \ldots, x_m]$ is the Schur function labeled by λ in $x_1, \ldots, x_m$ and $s_{\lambda'}(y)$ is the Schur function labeled by λ' in $y_1, \ldots, y_n$ and the sum is over partitions λ such that λ has at most m rows and λ' has at most n rows. Before launching the general proof do some illustrative small and smallish examples.

(14) (The $\widehat{\mathfrak{sl}}_2$-crystal $B(\Lambda_0)$) A partition $\mu = (\mu_1, \mu_2, \ldots)$ is *p-regular* if μ satisfies:

$$\text{if } k \in \mathbb{Z}_{>0} \quad \text{then} \quad \mathrm{Card}\{j \mid \mu_j = k\} < p.$$

Let

$$\mathfrak{h}^*_\mathbb{Z} = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2\} \cong \mathbb{Z}^2 \qquad \text{and} \qquad \mathfrak{h}^*_\mathbb{R} = \mathbb{R}\text{-span}\{\varepsilon_1, \varepsilon_2\} \cong \mathbb{R}^2$$

and let $\omega_1 = \varepsilon_1$, $\Lambda_0 = \varepsilon_2$ and

$$\alpha_1 = 2\omega_1 \qquad \text{and} \qquad \alpha_0 = -\alpha_1.$$

For $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2$ let

$$\langle\lambda, \alpha_1^\vee\rangle = \lambda_1, \qquad \text{and} \qquad \langle\lambda, \alpha_0^\vee\rangle = \lambda_1 - \lambda_2.$$

Let p^+ be the straight line path to Λ_0 and let

$$B(\Lambda_0) \ \text{ be the crystal generated by the action of } \tilde{f}_0 \text{ and } \tilde{f}_1 \text{ on } p^+,$$

where $\tilde{f}_0$, $\tilde{f}_1$ (and $\tilde{e}_0$ and $\tilde{e}_1$) are the root operators on paths corresponding to α_0 and α_1. In other words,

$$B(\Lambda_0) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \mid k \in \mathbb{Z}_{\geqslant 0},\ i_1, \ldots, i_k \in \{0, 1\}\}.$$

Find a bijection between

$$B(\lambda_0) \longleftrightarrow \{\text{2-regular partitions}\}$$

(If the action of $\tilde{f}_0$ and $\tilde{f}_1$ is not clear enough to get going drawing some pictures of these paths then come ask me.) Before launching the general proof do some illustrative small and smallish examples.