<u>Representation Theory</u> Lecture 11 or 12, 21 August 2015 (1)
Univ. of Melbourne

Last time: Two GL_n-crystals

$$B^{\otimes k} = \{ p_{i_1} \otimes \cdots \otimes p_{i_k} \mid i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\} \}.$$

and $$B(\lambda) = \{ \text{column strict tableaux of shape } \lambda \}.$$

Let $x_1, x_2, \ldots, x_n$ be variables. Then

$$\mathrm{char}(B^{\otimes k}) = \mathrm{char}(B)^k = (x_1 + \cdots + x_n)^k \quad \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

since $B = \{p_1, \ldots, p_n\}$ and p_i is the straight line path to ε_i. The <u>Schur function</u> is.

$$s_\lambda = \mathrm{char}(B(\lambda)) = \sum_{p \in B(\lambda)} x^{p(1)} = \sum_{p \in B(\lambda)} x_1^{\#1\text{'s in } p} x_2^{\#2\text{'s in } p} \cdots x_n^{\#n\text{'s in } p}.$$

<u>Theorem</u> (The Weyl character formula)
Let $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$ be a partition. Then

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$$

$$= \frac{\sum_{w \in S_n} \det(w)\, w(x_1^{\lambda_1 + n - 1} x_2^{\lambda_2 + n - 2} \cdots x_{n-1}^{\lambda_{n-1} + 1} x_n^{\lambda_n})}{\sum_{w \in S_n} \det(w)\, w(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})}$$

where $w x_i = x_{w(i)}$ and $w(fg) = (wf)(wg)$ for $w \in S_n$, $f, g \in \mathbb{C}[x_1, \ldots, x_n]$

Proposition Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ be a partition.
Then

$$B(\lambda) \otimes B \simeq \bigoplus_{\mu/\lambda = \square} B(\mu),$$

where the sum is over partitions μ obtained from λ by adding a box.

Corollary $s_\lambda(x_1 + \cdots + x_n) = \sum_{\mu/\lambda} s_\mu$.

The Weyl denominator formula

$$\det(x_i^{n-j}) = \det \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & & & & \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix}$$

$$= \prod_{i<j} (x_i - x_j).$$

Theorem

(a) $(x_1 + \cdots + x_n)^k = \sum_{\lambda \vdash k} f^\lambda s_\lambda$

where f^λ = (# of standard tableaux of shape λ)

(b) $n! = \sum_{\lambda \vdash k} (f^\lambda)^2$.