Five "equivalences" of categories

(1) Passing to the tangent space.

$$\begin{Bmatrix}\text{groups with a}\\ \text{tangent space}\end{Bmatrix} \longrightarrow \{\text{Lie algebras}\}$$

$$G \longmapsto \mathfrak{g} = \mathrm{Lie}(G) = T_1(G)$$

$$\varphi : G \to H \longmapsto d\varphi : \mathfrak{g} \to \mathfrak{h}.$$

where

$$T_1(G) = \{\text{tangent vectors to } G \text{ at } 1\} \qquad \xi_1 : C^\infty(G) \to \mathbb{R}$$

$$\updownarrow$$

$$\begin{Bmatrix}\text{one parameter subgroups}\\ \text{of } G\end{Bmatrix} \qquad \gamma : \mathbb{R} \to G$$

$$\updownarrow$$

$$\begin{Bmatrix}\text{left invariant vector}\\ \text{fields on } G\end{Bmatrix} \qquad \xi : C^\infty(G) \to C^\infty(G)$$

where the bijections are given by

$$\xi_1(f) = (\xi f)(1) \quad \text{and} \quad \xi(f) = \left(\frac{d}{dt} f(\gamma(t))\right)\Bigg|_{t=0}.$$

If $\varphi: G \to H$ then $d\varphi: \mathfrak{g} \to \mathfrak{h}$ is given by

$$d\varphi(\xi_1) = \xi_1 \circ \varphi^*; \quad d\varphi(\xi) = \xi \circ \varphi^*, \quad d\varphi(\gamma) = \varphi \circ \gamma$$

where $\varphi^*: C^\infty(H) \to C^\infty(G)$ is the morphism on rings of functions corresponding to φ.

G is recovered from $\mathfrak{g}$

by the <u>exponential</u> <u>map</u> $\exp: \mathfrak{g} \to G$
$tX \mapsto e^{tX}$

where $e^{tX} = \gamma(t)$ if γ is the one-parameter subgroup corresponding to X.

(12) <u>"Weyl's unitary trick" and Tannaka-Krein.</u>

$$\left\{\begin{matrix}\text{Hopf}\\ \text{algebras}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{algebraic}\\ \text{groups}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{compact}\\ \text{Lie groups}\end{matrix}\right\}$$

$$\mathcal{O}_G \longmapsto G \longmapsto K$$

where $\mathcal{O}_G$ is the ring of functions of G

K is the maximal compact subgroup of G

K is recovered from $\mathcal{O}_G$ as

$K = \{ g \in \mathcal{O}_G \mid \Delta(g) = g \otimes g \}$ the group like elements in $\mathcal{O}_G$

and $\mathcal{O}_G$ is the ring of coordinate functions of finite dimensional representations of K.

(3) <u>Enveloping algebras and differential operators</u>

$$\begin{array}{ccc} \left\{\begin{matrix}\text{Lie}\\ \text{algebras}\end{matrix}\right\} & \xleftrightarrow{\;U\;} & \left\{\begin{matrix}\text{associative}\\ \text{algebras}\end{matrix}\right\} \\ \mathfrak{g} & \longmapsto & U\mathfrak{g} \end{array}$$

where $U\mathfrak{g}$ is the <u>enveloping</u> <u>algebra</u> of $\mathfrak{g}$, the associative algebra generated by the vectorspace $\mathfrak{g}$ with relations

$$xy - yx = [x,y], \quad \text{for } x,y \in \mathfrak{g}.$$

The functor U is the left adjoint to the functor

$$\begin{array}{ccc} \left\{\begin{matrix}\text{associative}\\ \text{algebras}\end{matrix}\right\} & \xrightarrow{\;L\;} & \left\{\begin{matrix}\text{Lie}\\ \text{algebras}\end{matrix}\right\} \\ A & \longmapsto & L(A) \end{array}$$

where $L(A) = A$ with bracket given by

$$[a_1, a_2] = a_1 a_2 - a_2 a_1, \quad \text{for } a_1, a_2 \in A.$$

U is the <u>left</u> <u>adjoint</u> to L means

$$\mathrm{Hom}_{alg}(U\mathfrak{g}, A) = \mathrm{Hom}_{Lie}(\mathfrak{g}, L(A))$$

for all Lie algebras $\mathfrak{g}$ and all assoc. algebras A.

<u>Remark</u>: $U\mathfrak{g}$ "=" $\mathcal{O}_G^*$ (the dual of $\mathcal{O}_G$).

(4) Chevalley groups

$$\left\{\begin{matrix}\text{reductive}\\ \text{algebraic groups}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-reflection}\\ \text{groups}\end{matrix}\right\}$$

$$G \longmapsto (\mathfrak{h}_{\mathbb{Z}}, W_0)$$

where, if T is a maximal torus in G then

$$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T), \qquad \mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times)$$

and

$$W_0 = N(T)/T, \quad \text{where } N(T) \text{ is the normalizer of } T \text{ in } G.$$

The group G is recovered from $(\mathfrak{h}_{\mathbb{Z}}, W_0)$ as the group presented by generators

$$x_\alpha(c),\ x_{-\alpha}(c),\ h_\lambda(t), \quad \text{for } \alpha \in R^+,\ c \in \mathbb{C}, \quad \lambda \in \mathfrak{h}_{\mathbb{Z}},\ t \in \mathbb{C}^\times$$

with relations

$$x_\alpha(c_1 + c_2) = x_\alpha(c_1)\, x_\alpha(c_2), \quad \text{etc.}$$

(5) p-compact groups

$$\{p\text{-compact groups}\} \longleftrightarrow \{\mathbb{Z}_p\text{-reflection groups}\}$$

$$BG \longleftarrow (\mathfrak{h}_{\mathbb{Z}_p}, W_0)$$