Representation Theory, Lecture 23, 17 September 2015

G a connected reductive linear algebraic group.

B a Borel subgroup.

G/B is the <u>flag variety</u>.

<u>Example</u> $G = GL_n(\mathbb{C})$ and $B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \subseteq G$.

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ (by matrix multiplication).

If $V \subseteq \mathbb{C}^n$ is a subspace with $\dim V = k$

then $gV = \{gv \mid v \in V\}$ is a subspace of dimension k.

A <u>flag</u> on $\mathbb{C}^n$ is a sequence of subspaces

$$0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n \text{ with } \dim V_i = i.$$

(If $V \subseteq W \subseteq \mathbb{C}^n$ are subspaces then $gV \subseteq gW \subseteq \mathbb{C}^n$).

Let $e_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots 0)^t$ and

$$F = (0 \subseteq \operatorname{span}\{e_1\} \subseteq \operatorname{span}\{e_1, e_2\} \subseteq \cdots \subseteq \operatorname{span}\{e_1, \ldots, e_n\})$$

Then

$$\begin{array}{ccc} \left\{\begin{matrix}\text{flags in} \\ \mathbb{C}^n\end{matrix}\right\} & \longleftrightarrow & G/B \\ gF & \longmapsfrom & gB \end{array} \quad \text{is a bijection}$$

(2) Let $1 \le i_1 < i_2 < \cdots < i_\ell < n$.

~~Let~~ A partial flag of type $(i_1, \dots, i_\ell)$ is a sequence of subspaces

$$0 \subseteq V_{i_1} \subseteq V_{i_2} \subseteq \cdots \subseteq V_{i_\ell} \subseteq \mathbb{C}^n \text{ with } \dim(V_{i_k}) = i_k.$$

Let

$$F_{i_1, \dots, i_\ell} = (0 \subseteq \mathrm{span}\{e_1, \dots, e_{i_1}\} \subseteq \mathrm{span}\{e_1, \dots, e_{i_2}\} \subseteq \cdots \subseteq \mathbb{C}^n).$$

Then let

$$P = \mathrm{stab}(F_{i_1 \cdots i_\ell}) = \left\{ \begin{array}{l} i_1\{ \\ i_2\{ \\ i_3\{ \\ \end{array} \begin{pmatrix} \boxed{} & & * & \\ & \boxed{} & & \\ & & \boxed{} & \\ 0 & & & \ddots \end{pmatrix} \right\}$$

~~and~~ Then

$$\left\{ \begin{array}{l} \text{partial flags} \\ \text{of type} \\ i_1, \dots, i_\ell \end{array} \right\} \longleftrightarrow G/P$$

$$g F_{i_1 \cdots i_\ell} \longleftarrow\!\shortmid \; gP$$

Let $1 \le k < n$.

(3) A <u>k-subspace</u> of $\mathbb{C}^n$ is a subspace

$$(0 \subseteq V \subseteq \mathbb{C}^n) \text{ with } \dim(V) = k.$$

Let

$$F_k = (0 \subseteq \mathrm{span}\{e_1, \dots, e_k\} \subseteq \mathbb{C}^n)$$

Then

$$P = \mathrm{stab}(F_k) = \left\{ \begin{array}{l} k\{ \\ \end{array} \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$$

and
$$\left\{\begin{matrix}\text{subspaces of}\\ \text{dim. } k\end{matrix}\right\} \longleftrightarrow G/P$$
$$gF_k \longleftarrow\!\shortmid\; gP$$

$\left\{\begin{matrix}\text{subspaces of}\\ \text{dim } k\end{matrix}\right\}$ is the <u>Grassmannian</u> of "k-planes" on $\mathbb{C}^n$.

(4) A <u>line</u> on $\mathbb{C}^n$ is a 1-dimensional subspace of $\mathbb{C}^n$ $(0 \subseteq V \subseteq \mathbb{C}^n)$ with $\dim V = 1$.

Let
$$F_1 = (0 \subseteq \operatorname{span}\{e_1\} \subseteq \mathbb{C}^n)$$
Let
$$P = \operatorname{Stab}(F_1) = \left\{ \left(\begin{array}{c|c} * & * \\ 0 & * \end{array} \right) \right\}$$

Then
$$\mathbb{P}^{n-1}(\mathbb{C}) = \left\{\begin{matrix}\text{lines in}\\ \mathbb{C}^n\end{matrix}\right\} \xleftarrow{\sim} G/P \xrightarrow{\sim} \frac{\mathbb{C}^n - \{0\}}{\langle (\lambda a_1, \dots, \lambda a_n) = (a_1, \dots, a_n) \text{ for } \lambda \in \mathbb{C}^\times \rangle}$$
$$gF_1 \longleftarrow\!\shortmid\; gP \longmapsto [g_{11}, g_{21}, \dots, g_{n1}]$$

$\left\{\begin{matrix}\text{lines}\\ \text{on } \mathbb{C}^n\end{matrix}\right\}$ is <u>projective space</u> $\mathbb{P}^{n-1}(\mathbb{C})$.