Representation Theory Lecture 26, 24 September 2015
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Let $\mathfrak{g}$ be a Lie algebra with basis $\{x_1, \ldots, x_\ell\}$.

The <u>enveloping algebra of $\mathfrak{g}$</u> is the associative algebra $U\mathfrak{g}$ generated by $x_1, \ldots, x_\ell$ with relations

$$x_i x_j = x_j x_i + \sum_{k=1}^{\ell} c_{ij}^k x_k \quad \text{if} \quad [x_i, x_j] = \sum_{k=1}^{\ell} c_{ij}^k x_k \text{ in } \mathfrak{g}.$$

<u>Poincaré-Birkhoff-Witt theorem</u>

$U\mathfrak{g}$ has $\mathbb{C}$-basis

$$\{x_1^{m_1} x_2^{m_2} \cdots x_\ell^{m_\ell} \mid m_1, m_2, \ldots, m_\ell \in \mathbb{Z}_{\geq 0}\}.$$

<u>Example</u> $\mathfrak{sl}_2$ has basis e, f, h where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$U\mathfrak{sl}_2$ is generated by symbols e, f, h with relations

$$ef = fe + h, \quad he = eh + 2e, \quad fh = hf + 2f.$$

$U\mathfrak{sl}_2$ has basis

$$\{f^a h^b e^c \mid a, b, c \in \mathbb{Z}_{\geq 0}\}.$$

Let $\mathfrak{g}$ be a complex reductive Lie algebra.

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{\alpha}\right)$$

$$= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $e_{\beta_1}, e_{\beta_2}, \ldots, e_{\beta_N}$ be a basis of $\mathfrak{n}^+$

$h_1, h_2, \ldots, h_n$ be a basis of $\mathfrak{h}$.

$f_{\beta_1}, \ldots, f_{\beta_N}$ be a basis of $\mathfrak{n}^-$

Then

$U^- = U\mathfrak{n}^-$ has basis $\{ f_{\beta_1}^{m_1} f_{\beta_2}^{m_2} \cdots f_{\beta_N}^{m_N} \mid m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0}\}$

$U_0 = U\mathfrak{h}$ has basis $\{ h_1^{a_1} \cdots h_n^{a_n} \mid a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}\}$

$U^+ = U\mathfrak{n}^+$ has basis $\{ e_{\beta_1}^{n_1} e_{\beta_2}^{n_2} \cdots e_{\beta_N}^{n_N} \mid n_1, n_2, \ldots, n_N \in \mathbb{Z}_{\geq 0}\}$

and U has basis

$$\left\{ f_{\beta_1}^{m_1} f_{\beta_2}^{m_2} \cdots f_{\beta_N}^{m_N} h_1^{a_1} \cdots h_n^{a_n} e_{\beta_1}^{n_1} \cdots e_{\beta_N}^{n_N} \;\middle|\; \begin{array}{l} m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0} \\ a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0} \\ n_1, n_2, \ldots, n_N \in \mathbb{Z}_{\geq 0} \end{array} \right\}$$

so that $U = U^- U_0 U^+$. (triangular decomposition)

<u>Note</u>: $\mathbb{C}[x_1, \ldots, x_n]$ has basis $\{ x_1^{k_1} \cdots x_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \}$.

Induction is the left adjoint to Restriction

This means

$$(*) \qquad \mathrm{Hom}\left(\mathrm{Ind}_A^B(M), N\right) = \mathrm{Hom}\left(M, \mathrm{Res}_A^B(N)\right).$$

So

$$\begin{aligned} \mathrm{Res}_A^B : \{B\text{-modules}\} &\longrightarrow \{A\text{-modules}\} \\ N &\longmapsto N \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ind}_A^B : \{A\text{-modules}\} &\longrightarrow \{B\text{-modules}\} \\ M &\longmapsto \mathrm{Ind}_A^B(M) = B \otimes_A M. \end{aligned}$$

Note that $(*)$ is a universal property:

The <u>induced module $\mathrm{Ind}_A^B(M)$</u> is a B-module with an A-module homomorphism $\iota : M \to \mathrm{Ind}_A^B(M)$ such that if N is a B-module and

$\varphi : M \to N$ is an A-module homomorphism

then there exists a unique B-module homomorphism

$\psi : \mathrm{Ind}_A^B(M) \to N$ such that $\psi \circ \iota = \varphi$.

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{Ind}_A^B(M) \arrow[d, dashed, "\psi"] \\ & N \end{tikzcd}$$

Verma modules for sl_2

Let $\mathfrak{g} = sl_2$ with $\mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a+d=0 \right\}$

and $\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| a+d=0 \right\} = \mathbb{C}h.$

$\mathfrak{g} = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ with $R^+ = \{\alpha_1\}$ and

$\alpha_1 : \mathfrak{h} \to \mathbb{C}$, $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mapsto 2a$ so that $\alpha_1(h) = \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2$

~~Then~~ since $[h,e] = \alpha_1(h)e = 2e$.

Then $\mathfrak{h}^* = \mathrm{span}\{\omega_1\}$ where $\omega_1 : \mathfrak{h} \to \mathbb{C}$, $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mapsto a$ so that $\alpha_1 = 2\omega_1$

Let $\mu \in \mathfrak{h}^*$. So $\mu = c\omega_1$. Then $\mathbb{C}_\mu$ is the $\mathfrak{b}$-module

$\mathbb{C}_\mu = \mathbb{C}v^+$ with $hv^+ = cv^+$ and $ev^+ = 0$.

So $U\mathfrak{b} = \mathrm{span}\{h^a e^n \mid a,n \in \mathbb{Z}_{\geq 0}\}$ and $h^a e^n v^+ = 0$ if $n > 0$

and $h^a v^+ = c^a v^+$ for $a \in \mathbb{Z}_{\geq 0}$.

Then $U\mathfrak{g} = \mathrm{span}\{f^m h^a e^n \mid m,a,n \in \mathbb{Z}_{\geq 0}\} = U^- U_0 U^+ = U$

$M(\mu) = U\mathfrak{g} \otimes_{U\mathfrak{b}} v^+ = Uv^+ = U^- U_0 U^+ v^+ = U^- v^+$

has basis $\{f^m v^+ \mid m \in \mathbb{Z}_{\geq 0}\}$. The action of f in this basis is by the matrix

$$\rho(f) = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix}$$ where $\rho : U\mathfrak{g} \to \mathrm{End}(M(\mu))$.