Representation Theory, Lecture 32, 15 October 2015 ①

As $\mathfrak{h}$-modules,

$$\mathfrak{g} = \underbrace{\Big(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}\Big)} \oplus \mathfrak{h} \oplus \underbrace{\Big(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{\alpha}\Big)}$$

$$= \mathfrak{n}^- \oplus \underbrace{\mathfrak{h} \oplus \mathfrak{n}^+}$$

$$= \mathfrak{n}^- \oplus \mathfrak{b}$$

G is generated by $\quad x_{-\alpha}(c),\ h_{\lambda^\vee}(d),\ x_\alpha(c)$

$\cup|$

B is generated by $\quad h_{\lambda^\vee}(d),\ x_\alpha(c)$

$\cup|$

T is generated by $\quad h_{\lambda^\vee}(d)$

and our favourite example is

$$GL_n(\mathbb{F}) \supseteq \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \supseteq \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$$

with $\quad R^+ = \{\varepsilon_i - \varepsilon_j \mid i,j\in\{1,2,\ldots,n\} \text{ with } i<j\}$

$$x_{\varepsilon_i-\varepsilon_j}(c) = \begin{array}{c c} & j \\ i & \begin{pmatrix} 1 & & & \\ & \ddots & c & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array} \qquad x_{-(\varepsilon_i-\varepsilon_j)}(c) = \begin{array}{c c} & i \\ j & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & c & \ddots & \\ & & & 1 \end{pmatrix} \end{array}$$

$$h_{\varepsilon_i^\vee}(d) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & d & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$$

Define

$$n_\alpha(d) = x_\alpha(d)\, x_{-\alpha}(-d^{-1})\, x_\alpha(d)$$

$$n_\alpha = n_\alpha(1)$$

$$n_\alpha(d) = h_{\alpha^\vee}(d)\, n_\alpha.$$

$$N = \langle n_\alpha(d), h_{\lambda^\vee}(d) \mid d \in \mathbb{F}^\times \rangle$$

and
$$N \longrightarrow N/T = W_0, \quad \text{with } W_0 \text{ the } \underline{\text{Weyl group}}$$
$$n_\alpha(d) \longmapsto s_\alpha.$$

The <u>~~root~~ subgroups</u> are

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{F}\} \quad \text{and} \quad \mathcal{X}_{-\alpha} = \{x_{-\alpha}(c) \mid c \in \mathbb{F}\}.$$

Note that, since ~~x_α is the~~ $x_\alpha(c) = \exp(cX_\alpha)$ with $X_\alpha \in \mathfrak{g}_\alpha$.

$$x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1+c_2) \quad \text{and} \quad x_{-\alpha}(c_1)x_{-\alpha}(c_2) = x_{-\alpha}(c_1+c_2).$$

More relations:

$$h_{\lambda^\vee}(d)\,h_{\gamma^\vee}(d) = h_{\lambda^\vee+\gamma^\vee}(d) \quad \text{and}$$
$$h_{\lambda^\vee}(d_1)\,h_{\lambda^\vee}(d_2) = h_{\lambda^\vee}(d_1d_2).$$

and (since $h_{\lambda^\vee}(d) = \exp(d\lambda^\vee)$ with $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$).

$$h_{\lambda^\vee}(d)\,x_\alpha(c)\,h_{\lambda^\vee}(d)^{-1} = x_\alpha(d^{\langle\lambda^\vee,\alpha\rangle}c)$$

and
$$n_\alpha\, h_{\lambda^\vee}(d)\, n_\alpha^{-1} = h_{s_\alpha\lambda^\vee}(d)$$
$$n_\alpha\, x_\beta(c)\, n_\alpha^{-1} = x_{s_\alpha\beta}(c).$$

and finally, if $\alpha, \beta \in R^+$

$$x_\alpha(c_1)x_\beta(c_2) = x_\beta(c_2)x_\alpha(c_1)\,x_{\alpha+\beta}(?)\,x_{2\alpha+\beta}(?)\,x_{\alpha+2\beta}(?)\cdots$$

$GL_2(\mathbb{F}) = G$

$$x_{\alpha_1}(c) = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix} \qquad x_{-\alpha_1}(c) = \begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}$$

$$n_{\alpha_1}(c) = x_{\alpha_1}(c)\,x_{-\alpha_1}(-c^{-1})\,x_{\alpha_1}(c) = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \\ -c^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & c\\ & 1\end{pmatrix}$$

$$= \begin{pmatrix}0 & c\\ -c^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & c\\ -c^{-1} & 0\end{pmatrix} = \begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

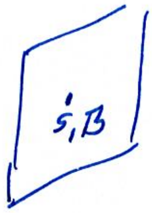
$= h_{\alpha_1^\vee}(c)\,n_{\alpha_1}$, where $n_{\alpha_1} = n_{\alpha_1}(1)$ and $h_{\alpha_1^\vee}(c) = n_{\alpha_1}(c)\,n_{\alpha_1}^{-1}$.

$$B = \left\{\begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \in G\right\} = \langle h_{\alpha_1^\vee}(d),\ x_{\alpha_1}(c) \mid d \in \mathbb{C}^\times,\ c \in \mathbb{C}\rangle.$$

Then $G = B \sqcup Bs_1B$ and

$$Bs_1B = \{x_\alpha(c)\,n_{\alpha_1}^{-1}B \mid c \in \mathbb{C}\} = \left\{\begin{pmatrix}c & -1\\ 1 & 0\end{pmatrix}B \;\middle|\; c \in \mathbb{C}\right\}$$

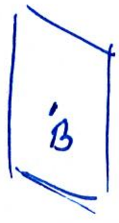
so that

$G/B = pt \sqcup \mathbb{C} = \dot{B} \sqcup$  $=$  $= \mathbb{P}^1$

especially since $\begin{pmatrix}c & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c^{-1} & 1\\ 0 & c\end{pmatrix} = \begin{pmatrix}1 & 0\\ c^{-1} & 1\end{pmatrix}$ then

$$\begin{pmatrix}c & -1\\ 1 & 0\end{pmatrix}B = \begin{pmatrix}1 & 0\\ c^{-1} & 1\end{pmatrix}B \quad \text{if } c \neq 0.$$

So

$G/B = \left\{\begin{pmatrix}1 & 0\\ d & 1\end{pmatrix}B \;\middle|\; d \in \mathbb{C}\right\} \sqcup s_1B =$  $\sqcup\ \dot{s_1B} =$