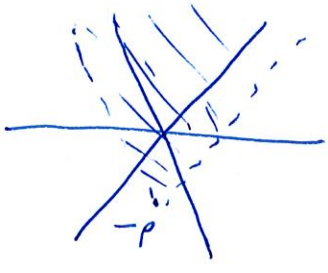
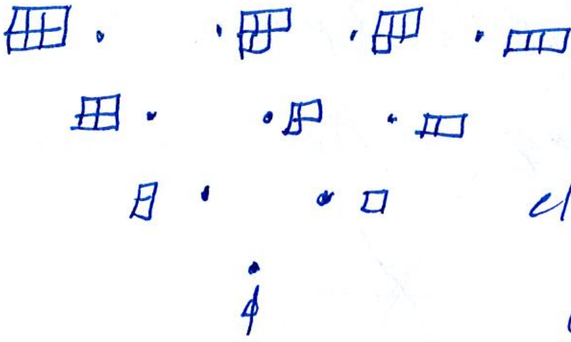
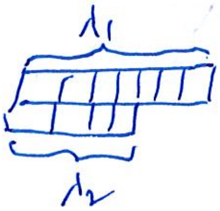
Representation Theory Lecture 4, 14.08.2015
Univ. of Melbourne

Example $B = \{\nearrow, \leftarrow, \searrow\} = \{p_1, p_2, p_3\}$





are the elements of $(\mathfrak{h}_{\mathbb{Z}}^*)^+$



$= \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2$ with $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_2 + \varepsilon_1$

B has hw path $\nearrow$ with endpoint (1).

$B\otimes B$ has hw paths $\nearrow\nearrow$ and $\nearrow\leftarrow$ with ends (2) and $(1,1)$

$B((2))\otimes B$ has hw paths $\nearrow\nearrow\nearrow$ and $\nearrow\nearrow\leftarrow$ with ends (3) and $(2,1)$

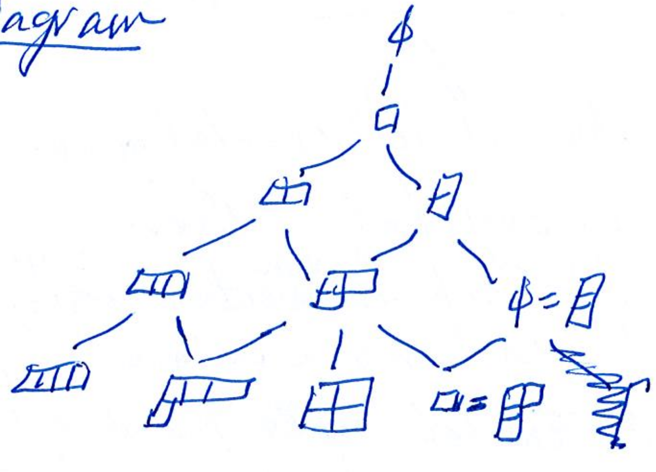
$B((1,1))\otimes B$ has hw paths $\nearrow\leftarrow\nearrow$ and $\nearrow\leftarrow\searrow$ with ends $(2,1)$ and ϕ

$B((3))\otimes B$ has hw paths $\nearrow\nearrow\nearrow\nearrow$ and $\nearrow\nearrow\nearrow\leftarrow$ with ends (4) and $(3,1)$

$B((2,1))\otimes B$ has hw paths $\nearrow\leftarrow\nearrow\nearrow$ and $\nearrow\leftarrow\nearrow\leftarrow$ and $\nearrow\leftarrow\nearrow\searrow$ with ends

$(3,1)$ and $(2,2)$ and (1)

Bratelli diagram



$B = B((1))$

$B \otimes B = B((2)) \oplus B((1,1))$

$B \otimes B \otimes B = B((3)) \oplus 2B((2,1)) \oplus B((1,1,1))$

$B \otimes B \otimes B \otimes B = B((4)) \oplus 3B((3,1)) \oplus 2B((2,2)) \oplus 3B((2,1,1))$

Next:

$\mathrm{char}(B) = x_1 + x_2 + x_3$, where $x_1 = X^{\varepsilon_1}$, $x_2 = X^{\varepsilon_2}$, $x_3 = X^{\varepsilon_3}$.

$\mathrm{char}(B((2))) = x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 + x_2^2 + x_3^2$

$\mathrm{char}(B((1,1))) = x_1x_2 + x_1x_3 + x_2x_3$

$\mathrm{char}(B((3))) = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2$
$+ x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3$

$\mathrm{char}(B((2,1))) = x_1^2x_2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_2x_3^2$

$\mathrm{char}(B((1,1,1))) = x_1x_2x_3.$

Note that

$(x_1+x_2+x_3) = s_{(1)}$

$(x_1+x_2+x_3)^2 = s_{(2)} + s_{(1,1)}$

$(x_1+x_2+x_3)^3 = s_{(3)} + 2s_{(2,1)} + s_{(1,1,1)}$

$(x_1+x_2+x_3)^4 = s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)}$

<u>GL_n data</u>: The symmetric group

$$W = S_n \quad \text{acts on} \quad \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$$

by permuting $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ so that

$$w\varepsilon_i = \varepsilon_{w(i)} \quad \text{for } w \in S_n, \; i \in \{1, \ldots, n\}.$$

Then S_n is generated by reflections in the hyperplanes

$$\mathfrak{h}^{\alpha_i^\vee} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R}, \lambda_i = \lambda_{i+1}\}$$

in $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}} = \mathbb{R}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n\} = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R}\}$.

The chamber is

$$C = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R}, \lambda_i > \lambda_{i+1}\}$$

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + 2\varepsilon_{n-2} + 1\cdot\varepsilon_{n-1} + 0\cdot\varepsilon_n$$

and if $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n$ then

$$\langle \lambda, \alpha_i^\vee \rangle = \langle \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \alpha_i^\vee \rangle = \lambda_i - \lambda_{i+1}$$

is the "distance from λ to the $\mathfrak{h}^{\alpha_i^\vee}$ wall".

Note that

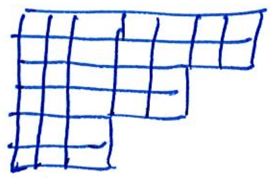
$$(C - \rho) \cap \mathfrak{h}^*_{\mathbb{Z}} = \left\{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \;\middle|\; \begin{matrix}\lambda_i \in \mathbb{Z}, \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\end{matrix}\right\}$$

Let p_i be the straight line path to ε_i

Then
$B = B(p_1)$ has crystal graph

$$p_1 \xrightarrow{1} p_2 \xrightarrow{2} p_3 \xrightarrow{3} \cdots \longrightarrow p_{n-1} \xrightarrow{n-1} p_n$$

Let $\lambda = \lambda_1 \varepsilon_1 + \cdots \lambda_n \varepsilon_n$ with $\lambda_i \geq \lambda_{i+1}$ and $\lambda_n \geq 0$.

$\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n =$ 

with λ_i = (number of boxes in row i).

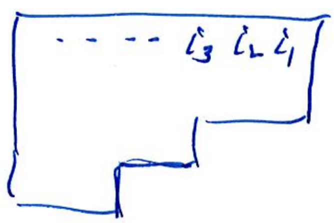
Let

$$p_\lambda = \underbrace{p_1 \otimes p_1 \otimes \cdots \otimes p_1}_{\lambda_1 \text{ times}} \otimes \underbrace{p_2 \otimes \cdots \otimes p_2}_{\lambda_2 \text{ times}} \otimes \cdots \otimes \underbrace{p_n \otimes \cdots \otimes p_n}_{\lambda_n \text{ times}}.$$

<u>Then</u> p_λ is a highest weight path

in $B \otimes B \otimes \cdots \otimes B$ and

$$B(p_\lambda) \longleftrightarrow \left\{ \begin{array}{l} \text{column strict tableaux} \\ \text{of shape } \lambda \end{array} \right\}$$

$p_{i_1} p_{i_2} \cdots p_{i_k} \longmapsto$ 

(Arabic reading)