Murphy elements in diagram algebras

Manuela Mazzocco
mmazzocco2@wisc.edu

and

Arun Ram
ram@math.wisc.edu

Department of Mathematics
University of Wisconsin
Madison, WI 53706

1 Temperley-Lieb algebras

1.1 Generators and relations

The *Temperley-Lieb algebra*, $\mathbb{C}T_k(n)$, is the algebra over $\mathbb{C}$ given by generators $E_1, E_2, ..., E_{k-1}$ and relations

$$\begin{aligned} &E_iE_j = E_jE_i, && \text{if } |i-j|>1, \\ &E_iE_{i\pm1}E_i = E_i, && \text{and} \\ &E_i^2 = nE_i. && \end{aligned}$$

If

$$[2] = q + q^{-1} = n \qquad \text{then} \qquad q = \frac{1}{2}(n+\sqrt{n^2-4}), \quad q^{-1} = \frac{1}{2}(n-\sqrt{(n^2-4}),$$

since $q^2 - nq + 1 = 0$. Then

$$[k] = \frac{q^k - q^{-k}}{q-q^{-1}} = \frac{1}{2^{k-1}} \sum_{m=1}^{(k+1)/2} \binom{k}{2m-1} n^{k-2m+1}(n^2-4)^{m-1}.$$

The problem with this expression is that it is not clear that $[k]$ is a polynomial in n with integer coefficients (which alternate in sign?).

The *Iwahori-Hecke algebra* $H_k(q)$ is the algebra over $\mathbb{C}$ with generators $T_1, T_2, ..., T_k - 1$ and relations

$$\begin{aligned} &T_iT_j = T_jT_i, && \text{if } |i-j|>1, \\ &T_iT_{i\pm1}T_i = T_{i+1}T_iT_{i+1}, && \text{if } 2 \le i \le k-1, \\ &T_i^2 = (q-q^{-1})T_i + 1. && \end{aligned}$$

There is a surjective algebra homomorphism

$$\varphi\colon H_k(q) \longrightarrow T_k(n) \qquad \text{given by} \qquad \varphi(T_i) = E_i - q^{-1} \quad \text{and} \quad \varphi(q+q^{-1}) = n.$$

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with

$$\ker\varphi = \langle T_iT_{i+1}T_i + T_iT_{i+1} + T_{i+1}T_i + T_i + T_{i+1} + 1\rangle$$

Composing with the surjective homomorphism

$$\begin{array}{ccc} \tilde{H}_k(q) & \longrightarrow & H_k(q) \\ X^{\varepsilon_i} & \longmapsto & T_{i-1}\cdots T_2T_1^1T_2\cdots T_{i-1} \\ T_i & \longmapsto & T_i \end{array}$$

1.2 Murphy elements

Let us write

$$T_i = E_i - q^{-1}, \qquad \text{so that} \quad X^{\varepsilon_1} = 1, \quad \text{and} \quad X^{\varepsilon_i} = T_{i-1}X^{\varepsilon_{i-1}}T_{i-1}$$

in the Temperley-Lieb algebra. Then define $m_1, \ldots, m_k$ by

$$m_1 = 0 \qquad \text{and} \qquad (q-q^{-1})m_j = q^{i-2}X^{\varepsilon_i}) - q^{i-4}X^{\varepsilon_{i-1}} \quad \text{for } 2 \leq i \leq k.$$

Soling for X^{ε_i} in terms of the m_i gives

$$X^{\varepsilon_i} = (q-q^{-1})(q^{-(i-2)}m_i + q^{-(i-2+1)}m_{i-1} + \cdots + q^{-(2i-4)}m_2) + q^{-2(i-1)},$$

from which one obtains

$$q^{(k-2)}(X^{\varepsilon_1} + X^{\varepsilon_2} + \cdots + X^{\varepsilon_k}) - q[k] = (q-q^{-1})(m_k + [2]m_{k-1} + \cdots + [k-1]m_2).$$

Using the definition of X^{ε_i} and substituting for $X^{\varepsilon_{i-1}}$ in terms of the m_i gives

$$\begin{aligned}
(q-q^{-1})m_i &= q^{i-2}X^{\varepsilon_i} - q^{i-4}X^{\varepsilon_{i-1}} \\
&= q^{i-2}(E_{i-1}-q^{-1})X^{\varepsilon_{i-1}}(E_{i-1}-q^{-1}) - q^{i-4}X^{\varepsilon_{i-1}} \\
&= q^{i-2}E_{i-1}X^{\varepsilon_{i-1}}E_{i-1} - q^{i-3}(E_{i-1}X^{\varepsilon_{i-1}} + X^{\varepsilon_{i-1}}E_{i-1}) \\
&= q^{i-2}E_{i-1}\big((q-q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)E_{i-1} \\
&\quad - q^{i-3}E_{i-1}\big((q-q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&\quad - q^{i-3}\big((q-q^{-1})(q^{-(i-3)}m_i + q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)E_{i-1} \\
&= q^{i-2}(q-q^{-1})q^{-(i-3)}E_{i-1}m_{i-1}E_{i-1} - q^{i-3}(q-q^{-1})q^{-(i-3)}(E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
&\quad + q^{i-2}(q+q^{-1})E_{i-1}\big((q-q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&\quad - 2q^{i-3}E_{i-1}\big((q-q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big) \\
&= q^{i-2}(q-q^{-1})q^{-(i-3)}E_{i-1}m_{i-1}E_{i-1} - q^{i-3}(q-q^{-1})q^{-(i-3)}(E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
&\quad + q^{i-2}(q-q^{-1})E_{i-1}\big((q-q^{-1})(q^{-(i-3+1)}m_{i-1} + \cdots + q^{-(2i-6)}m_2) + q^{-2(i-2)}\big)
\end{aligned}$$

since E_{i-1} commutes with $m_2, m_3, \ldots, m_{i-1}$. Thus

$$\begin{aligned}
m_i = q^{-(i-2)}&E_{i-1} + qE_{i-1}m_{i-1}E_{i-1} - (E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\
&+ (q-q^{-1})(m_{i-2} + q^{-1}m_{i-3} + q^{-2}m_{i-4} + \cdots + q^{-(i-4)}m_2)E_{i-1}.
\end{aligned}$$

It seems to me that this formula provides the easiest way to compute m_i in terms of the Es. I would not be too worried about the coefficients of E_1E_4 and E_2E_4 in m_4 looking strange. One expects diagrams that are equal to their own flip to act a bit differently in m_k. Note also that

$$[3] - 1 = \frac{[4]}{[2]} \quad \text{and} \quad [3] + 1 = [2]^2,$$

so these are pretty nice q-versions of 2. Let's have a look at m_6 and see if we can get an induction going. It might help to categorize the terms according to what their flip is to see where the next level is coming from.

For n such that $\mathbb{C}T_k(n)$ is semisimple, the simple $T_k(n)$ are indexed by partitions in the set

$$\hat{T}_k = \{\lambda \vdash k \mid \lambda \text{ has at most two columns}\}.$$

The irreducible $\mathbb{C}T_k(n)$ modules have seminormal basis

$$\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$$

and

$$X^{\varepsilon_i} v_T = q^{2c(T(i))} v_T.$$

Since $c(T(i)) = c(T(i-1)) - 1$ if the boxes $T(i)$ and $T(i-1)$ are in the same column and $c(T(i)) + c(T(i-1)) = 3 - i$ if the boxes $T(i)$ and $T(i-1)$ are in different columns it follows that

$$m_i v_T = \frac{q^{i-2} q^{2c(T(i))} - q^{i-4)} q^{2c(T(i-1))}}{q - q^{-1}} = c_T(i) v_T,$$

where

$$c_T(i) = \begin{cases} 0, & \text{if } T(i) \text{ and } T(i-1) \text{ are in the same column,} \\ [i - 2 + 2c(T(i))], & \text{if } T(i) \text{ and } T(i-1) \text{ are in different columns.} \end{cases}$$

Now we want to define pseudomatrix units in $\mathbb{C}T_k(n)$ according to the left and right eigenspaces of the m_i. Let

$$p_{ST} \in L_S \cap R_T,$$

normalized so that the coefficients are in $\mathbb{Z}[n]$ with greatest common divisor 1. Then

$$\begin{aligned} p_{ST} p_{UV} &= \gamma_T \delta_{UV} p_{SV}, \\ p_{ST} &= \sum_{S^+, T^+} c_{S^+T^+} p_{S^+T^+}, \\ p_{ST} e_k p_{UV} &= \beta_{T^-} \delta_{T^-U^-} p_{S^+V^+}, \\ e_{k+1} p_{ST} e_{k+1} &= \varepsilon_{S^+T^+} \delta_{S(k)T(k)} p_{ST} e_{k+1} \end{aligned}$$

1.3 Examples

Lets start with generic n. First note that

$$[1] = 1, \qquad [2] = n, \qquad [3] = n^2 - 1, \qquad [4] = n(n^2 - 2).$$

There is an isomorphism

$$\begin{array}{rcl} \mathbb{C}T_1(n) & \longrightarrow & M_1(\mathbb{C}) \\ e_{11} = 1 & \longmapsto & (1) \end{array} \qquad \text{and} \qquad \mathcal{E} = (e_1 e_{11} e_1) = ([2] e_1).$$

There is an algebra isomorphism

$$\begin{array}{rcl} \mathbb{C}T_2(n) & \longrightarrow & M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \\ e_{12,12} = \frac{1}{[2]} e_{11} e_1 e_{11} = \frac{1}{[2]} e_1 & \longmapsto & \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \\ e_{\substack{1\,1 \\ 2'2}} = 1 - e_{12,12} = 1 - \frac{1}{[2]} e_1 & \longmapsto & \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \end{array}$$

Abusing notation by identifying $\mathbb{C}T_2(n)$ with $M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ with this isomorphism

$$e_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \qquad \text{and} \qquad m_2 = \begin{pmatrix} [1] & \\ & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbb{C}T_1 = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\}.$$

Then

$$\mathcal{E} = \begin{pmatrix} e_2 e_{12,12} e_2 & e_2 e_{12,12} e_{\substack{1\\2},\substack{1\\2}} e_2 \\ e_2 e_{\substack{1\\2},\substack{1\\2}} e_{12,12} e_2 & e_2 e_{\substack{1\\2},\substack{1\\2}} e_2 \end{pmatrix} = \begin{pmatrix} \frac{[1]}{[2]} e_2 & 0 \\ 0 & \frac{[3]}{[2]} e_2 \end{pmatrix}$$

There is an isomorphism

$$\mathbb{C}T_3(n) \longrightarrow M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$$

determined by the elements

$$\begin{pmatrix} e_{\substack{12\\3},\substack{12\\3}} & e_{\substack{12\\3},\substack{13\\2}} & \\ e_{\substack{13\\2},\substack{12\\3}} & e_{\substack{13\\2},\substack{13\\2}} & \\ & & e_{\substack{1\\2\\3},\substack{1\\2\\3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{[2]}{[1]}(e_{12,12} e_2 e_{12,12}) & \frac{[2]}{[3]}(e_{12,12} e_2 e_{\substack{1\\2},\substack{1\\2}}) & \\ n(e_{\substack{1\\2},\substack{1\\2}} e_2 e_{12,12}) & \frac{[2]}{[3]}(e_{\substack{1\\2},\substack{1\\2}} e_2 e_{\substack{1\\2},\substack{1\\2}}) & \\ & & 1 - e_{\substack{12\\3},\substack{12\\3}} - e_{\substack{13\\2},\substack{13\\2}} \end{pmatrix}$$

In this basis

$$e_1 = \begin{pmatrix} n & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} \frac{[1]}{[3]} & 1 & \\ [3] & \frac{[3]}{[2]} & \\ & & 0 \end{pmatrix}, \qquad m_1 = \begin{pmatrix} [2] & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} [1] & 0 & \\ 0 & [3] & \\ & & 0 \end{pmatrix},$$

and

$$\mathbb{C}T_1 = \left\{ \begin{pmatrix} a & 0 & \\ 0 & a & \\ & & a \end{pmatrix} \right\}, \qquad \text{and} \qquad \mathbb{C}T_2 = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & 0 & \\ 0 & a_2 & \\ & & a_2 \end{pmatrix} \right\}.$$

The special value $n = \pm 1$, i.e. when $[3] = 0$

We still have that $\mathbb{C}T_1 \cong M_1(\mathbb{C})$ with basis element $e_{11} = 1$. Then $\mathbb{C}T_2 \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ with

$$\begin{pmatrix} e_{12,12} & \\ & e_{\substack{1\\2},\substack{1\\2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}(e_{11} e_1 e_{11}) & \\ & 1 - e_{12,12} \end{pmatrix} = \begin{pmatrix} e_1 & \\ & 1 - e_1 \end{pmatrix}.$$

Then

$$e_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad m_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbb{C}T_1 = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\}.$$

inside $\mathbb{C}T_2$. Now let's remove denominators in the basis elements of $\mathbb{C}T_3$ and set

$$\begin{pmatrix} p_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ & & p_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\2\\3\end{smallmatrix}} \end{pmatrix}$$

$$= \begin{pmatrix} [2]e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & [2][3]e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ [2]e_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & [2][3]e_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ & & [3]e_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\2\\3\end{smallmatrix}} \end{pmatrix}$$

When $[3] = 0$,

$$p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} = p_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\2\\3\end{smallmatrix}},$$

and

$$\mathrm{Rad}(\mathbb{C}T_3) = \mathrm{span}\left\{ p_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}},\ p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}},\ p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} \right\}$$

and

$$\mathrm{Rad}^2(\mathbb{C}T_3) = \mathrm{span}\left\{ p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} \right\}$$

and we introduce

$$p^{(2)}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} = 1 - e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}},$$

so that

$$\frac{\mathbb{C}T_3}{R} \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$$

with basis

$$\begin{pmatrix} p_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & \\ & p^{(2)}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\end{smallmatrix}} \end{pmatrix} = \begin{pmatrix} e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} & \\ & 1 - e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\ \begin{smallmatrix}1&2\\3\end{smallmatrix}} \end{pmatrix}$$

In the basis

$$\begin{pmatrix} e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}} & p_{\begin{smallmatrix}1&2\\3\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&2\\3\end{smallmatrix}} & p_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}} & \\ & & p^{(2)}_{\begin{smallmatrix}1&3\\2\end{smallmatrix}\,\begin{smallmatrix}1&3\\2\end{smallmatrix}} \end{pmatrix}$$

we have

$$e_1 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 & 1 & \\ 1 & 1 & \\ & & 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 1 \end{pmatrix},$$

$$m_1 = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} -1 & 0 & \\ 0 & 1 & \\ & & 0 \end{pmatrix},$$

and

$$\mathbb{C}T_1 = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 & \\ 0 & 0 & \\ & & a \end{pmatrix} \right\} \qquad \text{and} \qquad \mathbb{C}T_2 = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & 0 & \\ 0 & 0 & \\ & & a_2 \end{pmatrix} \right\}$$

The special value $n = 0$, i.e. when $[2] = 0$.

We still have that $\mathbb{C}T_1 \cong M_1(\mathbb{C})$ with basis element $e_{11} = 1$. Remove denominators in the basis elements of $\mathbb{C}T_2$ and set

$$\begin{pmatrix} p_{12,12} & \\ & p_{\begin{smallmatrix}1\\2\end{smallmatrix}\,\begin{smallmatrix}1\\2\end{smallmatrix}} \end{pmatrix} = \begin{pmatrix} [2]e_{12,12} & \\ & [2]e_{\begin{smallmatrix}1\\2\end{smallmatrix}\,\begin{smallmatrix}1\\2\end{smallmatrix}} \end{pmatrix}$$

When $n = 0$ then

$$p_{12,12} = -p_{\begin{smallmatrix}1\\2\end{smallmatrix}\,\begin{smallmatrix}1\\2\end{smallmatrix}} \qquad \text{and} \qquad \mathrm{Rad}(\mathbb{C}T_2) = \mathrm{span}\{p_{12,12}\},$$

and we introduce

$$p^{(2)}_{12,12} = 1$$

so that $\mathbb{C}T_2/\mathrm{Rad}(\mathbb{C}T_2) \cong M_1(\mathbb{C})$ with basis element $p^{(1)}_{12,12}$. In the basis

$$\begin{pmatrix} p_{12,12} & \\ & p^{(2)}_{12,12} \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbb{C}T_1 = \left\{ \begin{pmatrix} 0 & \\ & a \end{pmatrix} \right\}$$

inside the algebra $\mathbb{C}T_2$. with respect to this basis there is a new matrix

$$\mathcal{E} = \begin{pmatrix} e_2 p_{12,12}^2 e_2 & e_2 p_{12,12} p_{12,12}^{(2)} e_2 \\ e_2 p_{12,12}^{(2)} p_{12,12} e_2 & e_2 (p_{12,12}^{(2)})^2 e_2 \end{pmatrix} = \begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is *not diagonal*. In $\mathbb{C}T_3$ the basis elements

$$\begin{pmatrix} p^{(2)}_{\substack{1\ 2\\3}\ \substack{1\ 2\\3}} & p_{\substack{1\ 2\\3}\ \substack{1\ 3\\2}} & \\ p^{(2)}_{\substack{1\ 3\\2}\ \substack{1\ 2\\3}} & p^{(2)}_{\substack{1\ 3\\2}\ \substack{1\ 3\\2}} & \\ & & p_{\substack{1\\2\\3}\ \substack{1\\2\\3}} \end{pmatrix}$$

$$= \begin{pmatrix} p_{12,12} e_2 p^{(2)}_{12,12} & p^{(2)}_{12,12} e_2 p^{(2)}_{12,12} & \\ p_{12,12} e_2 p_{12,12} & p_{12,12} e_2 p^{(2)}_{12,12} & \\ & & 1 - p^{(2)}_{\substack{1\ 2\\3}\ \substack{1\ 2\\3}} - p^{(2)}_{\substack{1\ 3\\2}\ \substack{1\ 3\\2}} \end{pmatrix}$$

form a set of matrix units. In this basis

$$e_1 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \qquad 1 = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{pmatrix},$$

$$m_2 = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \qquad m_3 = \begin{pmatrix} -1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix},$$

$$\mathbb{C}T_1 = \left\{ \begin{pmatrix} a_1 & 0 & \\ 0 & a_1 & \\ & & a_1 \end{pmatrix} \right\} \qquad \text{and} \qquad \mathbb{C}T_2 = \left\{ \begin{pmatrix} a_2 & \\ & a_1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a_1 & a_2 & \\ 0 & a_1 & \\ & & a_1 \end{pmatrix} \right\}.$$

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