

Conference on Higgs bundles and character 10.02.2024 ①
Brisbane

Varieties - Counting and polynomials: A. Ram

Catalan and Hodge.

work in progress with Lucas Cagnon, Laura Colmenarejo, Nat Thiem

Remark Theorem 8.2.3 proof in Oblomkov-Tsun

$$H^*(M_{n,m}) \cong H^*(\mathbb{A}^1/n)(n-m)$$

as RCR modules.

A n-periodic matrix is $a \in M_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{F}_q)$ such that

a) If $i, j \in \mathbb{Z}$ then $a_{i+n, j+n} = a_{ij}$.

b) If $i \in \mathbb{Z}$ then there exists $k \in \mathbb{Z}$ such that
if $j \in \mathbb{Z}_{\leq -1}$ then $a_{ij} = 0$

$\mathcal{G} = \{ \text{n-periodic matrices} \}$

$\cup$

$\mathcal{K} = \{ \text{block upper triangular} \}$

$\cup$

$\mathcal{L} = \{ \text{upper triangular} \}$

$G = \{ g \in \mathcal{G} \mid g \text{ is invertible} \}$

$\cup$

$K = \{ g \in G \mid g \in K \text{ and } g^{-1} \in K \}$

$\cup$

$I = \{ g \in G \mid g \in \mathcal{L} \text{ and } g^{-1} \in \mathcal{L} \}$

Let $\pi^{-1} = \sum_{i \in \mathbb{Z}} E_{i, i+1}$ with E_{ij} has 1 in (i, j) entry and 0 elsewhere.

Cells in affine Springer fibres

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$$Y_b^{-1}(\pi^{-m}) = \{g \in G(\mathbb{F}) \mid g^{-1} \pi^m g \in b\}$$

$$Y_k^{-1}(\pi^{-m}) = \{g \in G(\mathbb{F}) \mid g^{-1} \pi^m g \in k\}$$

Remark: $g \xrightarrow{\alpha} M_{n \times n}(\mathbb{F}_\ell((\ell)))$

$$\sum_{i \in \mathbb{Z}} E_{i, n, j+k+n+1} \longleftarrow 1 \in E_{ij}$$

A n-periodic permutation is $w \in \mathfrak{S}$ with exactly one nonzero entry in each row and each column and nonzero entries 1.

Identify w with a bijection $w: \mathbb{Z} \rightarrow \mathbb{Z}$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ let $t_\lambda = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

$$G = \bigsqcup_{w \in W} \mathbb{I} w \mathbb{I} \quad \text{and} \quad G = \bigsqcup_{\lambda \in \mathbb{Z}^n} \mathbb{I} t_\lambda K$$

$$\S \quad Y_b^{-1}(\pi^{-m}) = \bigsqcup_{w \in W} (Y_b^{-1}(\pi^{-m}) \cap \mathbb{I} w \mathbb{I}) \quad (1)$$

$$Y_k^{-1}(\pi^{-m}) = \bigsqcup_{\lambda \in \mathbb{Z}^n} (Y_k^{-1}(\pi^{-m}) \cap \mathbb{I} t_\lambda K) \quad (2)$$

Theorem

- (a) $W_{m/n} = \{w \in W \mid \tilde{w}'(i) < \tilde{w}'(itm)\}$
- (b) $\text{div}(w) = \{(i,j) \mid i < j < itm \text{ and } \tilde{w}'(i) > \tilde{w}'(j)\}$
- (c) $\text{area}(w) = \# \text{ boxes below the Dyck path of the corresponding parking function.}$

Parabolic packaging

Let $\mu_1, \dots, \mu_\ell \in \mathbb{Z}_{\geq 0}$ with $\mu_1 \geq \dots \geq \mu_\ell$, $\mu_1 + \dots + \mu_\ell = n$.

$\mathcal{M} = \{ \begin{smallmatrix} n\text{-periodic} \\ \text{matrices} \end{smallmatrix} \}$

or

$G = \{g \in \mathcal{M} \mid g \text{ is invertible}\}$

$\mathcal{P}_\mu = \{ \begin{smallmatrix} \mu\text{-block} \\ \text{upper triangular} \end{smallmatrix} \}$

$P_\mu = \{g \in G \mid g \in \mathcal{P}_\mu \text{ and } g^{-1} \in \mathcal{P}_\mu\}$

$$Y_\mu^{-1}(\pi^{-m}) = \{g P_\mu \in G/P_\mu \mid g^{-1} \pi^{-m} g \in P_\mu\}$$

the Hikita polynomial is

$$\tilde{H}_{m/n}(X; t, q) = \sum_{\mu} \text{Card}_{t,q}(Y_\mu^{-1}(\pi^{-m})) m_{\mu}(X)$$

where $m_{\mu}(X)$ is the monomial symmetric function.

Mikita Varagolov-Vassaret 11.02.2026 (3)
Theorem Gorsky-Mazin-Vazirani, Lusztig, Brisbane
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(a) # of nonempty cells in (1) $= m^n = \# \frac{m}{n}$ parking functions

(b) # of nonempty cells in (2) $= \frac{1}{m+n} \binom{m+n}{n} = \# \frac{m}{n}$ Dyck paths

Let $W_{\frac{m}{n}} = \{w \in W \mid Y_{\frac{1}{n}}^{-1}(\pi^{-m}) \cap I_w \neq \emptyset\}$

For $w \in W_{\frac{m}{n}}$ define $\text{dinv}(w)$ by

$$Y_{\frac{1}{n}}^{-1}(\pi^{-m}) \cap I_w \subseteq \#_2^{\text{dinv}(w)} \quad \begin{array}{l} \text{Gorsky} \\ \text{-Kokotov} \\ \text{-MacPherson} \end{array}$$

In $H^*(Y_{\frac{1}{n}}^{-1}(\pi^{-m}))$ the cohomological degree is

$$\text{Cdeg}([Y_{\frac{1}{n}}^{-1}(\pi^{-m}) \cap I_w]) = \text{dinv}(w)$$

The perverse degree is

$$\text{Pdeg}([Y_{\frac{1}{n}}^{-1}(\pi^{-m}) \cap I_w]) = \text{area}(w)$$

the Hodge polynomial is

$$\text{Card}_{\mathbb{Z}}(Y_{\frac{1}{n}}^{-1}(\pi^{-m})) = \sum_{w \in W_{\frac{m}{n}}} q^{\text{dinv}(w)} t^{\text{area}(w)}$$

$L_{m/n} = H^*(Y_B^{-1}(\pi^{-m}))$ is a RCA-module
where RCA is the rational Cherednik algebra.

Theorem $\tilde{H}_{m/n}(X; q, t) = \text{Fch}(L_{m/n})$

i.e. the bigraded character of $L_{m/n}$, as an S_n -module is $\tilde{H}_{m/n}$.

A parallel story

$n_1 \in \mathfrak{gl}_n$ nilpotent in Jordan form with block sizes $\lambda = (\lambda_1, \dots, \lambda_r)$

Extend to $n_1 \in \mathfrak{g}$ by n -periodicity.

The Modified Macdonald polynomial is

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \text{Card}_{q,t}(Y_{\beta_\mu}^{-1}(n_1)) m_\mu(X).$$

The integral form Macdonald polynomial is

$$J_\lambda(X; t, q) = \sum_{\nu} q^{-\text{Int}(\nu)} (q-1)^{|\nu|} \text{Card}_{q,t}(Y_{\beta_\nu}^{-1}(n_1)) m_\nu(X)$$

where

$$Y_{\beta_\nu}^{-1}(n_1) = \{g \in \mathcal{O}_I \mid g^{-1} n_1 g \in \text{Lev } I \cdot \mathfrak{h}\}$$

is a Lusztig variety

