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Calculus Lect (4)

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So $\theta(\vec{u},\vec{v}) = \frac{3\pi}{4}$

Unit vectors have length 1.

$\frac{1}{\|\vec{u}\|}\vec{u}$ has length 1.

In $\mathbb{R}^3$, if $\vec{u} = (-2,1,2)$ then

$$\frac{1}{\|\vec{u}\|}\cdot\vec{u} = \frac{1}{\sqrt{(-2)^2+1^2+2^2}}\cdot(-2,1,2)$$

$$= \frac{1}{\sqrt{9}}(-2,1,2) = \frac{1}{3}(-2,1,2) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

is a vector of length 1 in the same direction of $\vec{u}$.

Parallel vectors If $A = (2,0,-1)$ and $B = (1,2,-3)$ then

$$\vec{AB} = B - A = (1,2,-3) - (2,0,-1) = (-1,2,-2)$$

and $\|\vec{AB}\| = \sqrt{(-1)^2+2^2+(-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

and

$$\frac{1}{\|\vec{AB}\|}\vec{AB} = \frac{1}{3}(-1,2,-2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$$

and $\frac{-1}{\|\vec{AB}\|}\vec{AB} = \frac{-1}{3}(-1,2,-2) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

are unit vectors parallel to $\vec{AB}$.

Projections: Projection of $\vec{v}$ onto the $\vec{u}$ line

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \left\langle \vec{v}, \frac{\vec{u}}{\|\vec{u}\|} \right\rangle \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2}\vec{u}$$

If $\vec{u} = (3,1,-2)$ and $\vec{v} = (1,0,5)$

then

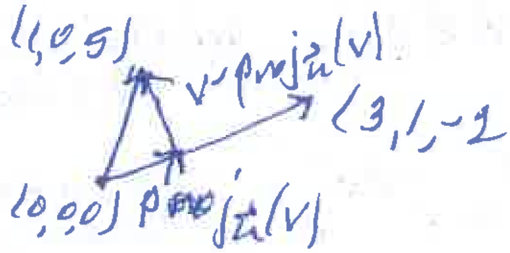
$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\langle (1,0,5), (3,1,-2) \rangle}{\|(3,1,-2)\|^2}(3,1,-2)$$

$$= \frac{1\cdot 3 + 0\cdot 1 + 5\cdot(-2)}{3^2+1^2+(-2)^2}(3,1,-2) = \frac{-7}{14}(3,1,-2)$$

$$= \left(\frac{-3}{2}, \frac{-1}{2}, 1\right)$$

and

$$\vec{v} - \mathrm{proj}_{\vec{u}}(\vec{v}) = (1,0,5) - \left(\frac{-3}{2}, \frac{-1}{2}, 1\right) = \left(\frac{5}{2}, \frac{1}{2}, 4\right)$$



$L = \{t\vec{u} \mid t \in \mathbb{R}\}$ is the line through the origin in the direction containing $\vec{u}$.

Then $\mathrm{proj}_{\vec{u}}(\vec{v})$ is the closest point to $\vec{v}$ on the line L.

<u>Span</u> the set of linear combinations (using addition and scalar multiplication).

$\{t\vec{u} \mid t \in \mathbb{R}\} = \text{span}\{\vec{u}\}$. Line containing $\vec{u}$

$\{c_1\vec{u} + c_2\vec{v} \mid c_1, c_2 \in \mathbb{R}\} = \text{span}\{\vec{u}, \vec{v}\}$ plane containing $\vec{u}$ and $\vec{v}$

<u>Problem</u> Is $(5,7) \in \text{span}\{(1,1),(2,3)\}$?

Is $(5,7) = c_1(1,1) + c_2(2,3)$ for some $c_1, c_2 \in \mathbb{R}$?

Do there exist $c_1, c_2 \in \mathbb{R}$ such that

$$5 = c_1 + 2c_2, \qquad c_1 = 5 - 2c_2$$
$$7 = c_1 + 3c_2 ? \qquad 7 = 5 - 2c_2 + 3c_2$$

So $c_1 = 5 - 2c_2$ So $c_1 = 5 - 2\cdot 2 = 1$
$2 = c_2$ $\quad c_2 = 2$.

So $(5,7) = 1\cdot(1,1) + 2\cdot(2,3)$

and $(5,7) \in \text{span}\{(1,1),(2,3)\}$.

In fact $(1,1) - \frac{1}{2}(2,3) = (0, -\frac{1}{2})$ so $-2(1,1) + (2,3) = (0,1)$

and $(1,1) - \frac{1}{3}(2,3) = (\frac{1}{3}, 0)$ so $3\cdot(1,1) - (2,3) = (1,0)$.

So $(1,0)$ and $(0,1)$ are in $\text{span}\{(1,1),(2,3)\}$.

So $\text{span}\{(1,0),(0,1)\} = \mathbb{R}^2$ is a subset of $\text{span}\{(1,1),(2,3)\}$.

11.08.2016 (3)
Calculus Lect,
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So $\mathbb{R}^2 = \text{span}\{(1,1),(2,3)\}$

If $(a,b) \in \mathbb{R}^2$ then

$$(a,b) = a(1,0) + b(0,1)$$
$$= a(3\cdot(1,1) - (2,3)) + b\cdot(-2(1,1) + (2,3))$$
$$= (3a-2b)\cdot(1,1) + (-a+b)\cdot(2,3).$$

<u>The "standard" basis of $\mathbb{R}^3$</u>

$$\hat{i} = (1,0,0), \quad \hat{j} = (0,1,0), \quad \hat{k} = (0,0,1)$$

Every vector $(a_1, a_2, a_3) \in \mathbb{R}^3$ is a linear combination of $\hat{i}, \hat{j}, \hat{k}$ (in a unique way)

$$(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$

For example

$$(5,2,1) = (5,0,0) + (0,2,0) + (0,0,1)$$
$$= 5\cdot(1,0,0) + 2\cdot(0,1,0) + (0,0,1)$$
$$= 5\hat{i} + 2\hat{j} + \hat{k}.$$

<u>Perpendicular vectors</u> $\vec{u}$ and $\vec{v}$ are perpendicular if $\theta(\vec{u},\vec{v}) = \frac{\pi}{2}$ i.e. if $\cos(\theta(\vec{u},\vec{v})) = \cos(\frac{\pi}{2}) = 0$

i.e. if $0 = \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\cdot\|\vec{v}\|}$ i.e. if $\langle \vec{u},\vec{v}\rangle = 0$