14.05.2024 3
Calculus Lect
A. Ram

Why do we like $\hat{i}, \hat{j}, \hat{k}$?

They are unit vectors

$$\langle \hat{i}, \hat{i} \rangle = \langle (1,0,0), (1,0,0) \rangle = 1+0+0 = 1$$

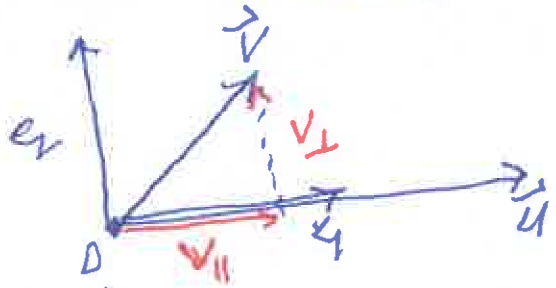
so $\quad \|\hat{i}\| = \sqrt{\langle \hat{i}, \hat{i} \rangle} = \sqrt{1} = 1.$

They are perpendicular

$$\langle \hat{i}, \hat{j} \rangle = \langle (1,0,0), (0,1,0) \rangle = 0\cdot 1 + 1\cdot 0 + 0\cdot 0 = 0.$$

So $\quad \langle \hat{i}, \hat{j} \rangle = 0.$

Let $\vec{u}, \vec{v}$ be vectors



$\vec{v} = \vec{v}_\parallel + \vec{v}_\perp \qquad$ Let $e_1 = \dfrac{\vec{u}}{\|\vec{u}\|}$.

e_1 is a unit vector parallel to $\vec{u}$.

Let e_2 be a unit vector perpendicular to $\vec{u}$

Well $v_\parallel = c_1 e_1$ and $v_\perp = c_2 e_2$. What are c_1 and c_2?

$$\vec{v} = v_\parallel + v_\perp = c_1 e_1 + c_2 e_2$$

and

$$\langle \vec{v}, \tfrac{1}{\|\vec{u}\|}\vec{u} \rangle = \langle \vec{v}, e_1 \rangle = \langle c_1 e_1 + c_2 e_2, e_1 \rangle$$
$$= c_1 \langle e_1, e_1 \rangle + c_2 \langle e_2, e_1 \rangle$$
$$= c_1 \cdot 1 + c_2 \cdot 0 = c_1$$

So $c_1 = \frac{1}{\|\vec{u}\|}\langle \vec{v}, \vec{u}\rangle$ and

$$v_{\parallel} = \frac{1}{\|\vec{u}\|}\langle \vec{v}, \vec{u}\rangle c_1 = \langle \vec{v}, \vec{u}\rangle \frac{1}{\|\vec{u}\|}\frac{1}{\|\vec{u}\|}\vec{u}$$

$$= \frac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}$$

So $\mathrm{proj}_{\vec{u}}(\vec{v}) = v_{\parallel} = \frac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}$.

We used the properties: If $\vec{u}, \vec{w}, \vec{z} \in \mathbb{R}^3$ and $c_1, c_2 \in \mathbb{R}$ then

(a) $\langle \vec{w}, \vec{u}\rangle = \langle \vec{u}, \vec{w}\rangle$

(b) $\langle c_1\vec{w} + c_2\vec{z}, \vec{u}\rangle = c_1\langle \vec{w}, \vec{u}\rangle + c_2\langle \vec{z}, \vec{u}\rangle$

(c) $\langle c\vec{w}, \vec{u}\rangle = c\langle \vec{w}, \vec{u}\rangle = \langle \vec{w}, c\vec{u}\rangle$

(b) should remind you of

$$\frac{d(c_1 f + c_2 g)}{dx} = c_1\frac{df}{dx} + c_2\frac{dg}{dx}$$

$$\int (c_1 f + c_2 g)\,dx = c_1\int f\,dx + c_2\int g\,dx.$$

Problem Show that if $\vec{u}, \vec{v} \in \mathbb{R}^3$ then

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle.$$

Solution: Let $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$. then

$$\langle \vec{u}, \vec{v} \rangle = \langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{and}$$
$$\langle \vec{v}, \vec{u} \rangle = \langle (b_1, b_2, b_3), (a_1, a_2, a_3) \rangle$$
$$= b_1 a_1 + b_2 a_2 + b_3 a_3 = a_1 b_1 + a_2 b_2 + a_3 b_3$$

since $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. //.

Problem Let $c_1, c_2 \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$.
Show that $\langle c_1\vec{u} + c_2\vec{v}, \vec{w} \rangle = c_1 \langle \vec{u}, \vec{w} \rangle + c_2 \langle \vec{v}, \vec{w} \rangle$.

Solution: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. The left hand side is

$$\langle c_1\vec{u} + c_2\vec{v}, \vec{w} \rangle = \langle c_1(u_1, u_2, u_3) + c_2(v_1, v_2, v_3), (w_1, w_2, w_3) \rangle$$
$$= \langle (c_1u_1, c_1u_2, c_1u_3) + (c_2v_1, c_2v_2, c_2v_3), (w_1, w_2, w_3) \rangle$$
$$= \langle (c_1u_1 + c_2v_1, c_1u_2 + c_2v_2, c_1u_3 + c_2v_3), (w_1, w_2, w_3) \rangle$$
$$= (c_1u_1 + c_2v_1) w_1 + (c_1u_2 + c_2v_2) w_2 + (c_1u_3 + c_2v_3) w_3$$

and ②

$$c_1\langle\vec{u},\vec{w}\rangle + c_2\langle\vec{v},\vec{w}\rangle = c_1\langle(u_1,u_2,u_3),(w_1,w_2,w_3)\rangle + c_2\langle(v_1,v_2,v_3),(w_1,w_2,w_3)\rangle$$

$$= c_1(u_1w_1+u_2w_2+u_3w_3) + c_2(v_1w_1+v_2w_2+v_3w_3)$$

$$= c_1u_1w_1 + c_2v_1w_1 + c_1u_2w_2 + c_2v_2w_2 + c_1u_3w_3 + c_2v_3w_3 = (c_1u_1+c_2v_1)w_1 + (c_1u_2+c_2v_2)w_2 + (c_1u_3+c_2v_3)w_3. \quad //$$

Properties of inner product If $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^3$ and $c_1,c_2\in\mathbb{R}$ then

(Symmetry)

$$\langle\vec{u},\vec{v}\rangle = \langle\vec{v},\vec{u}\rangle$$

(Linearity) $\langle c_1\vec{u}+c_2\vec{v},\vec{w}\rangle = c_1\langle\vec{u},\vec{w}\rangle + c_2\langle\vec{v},\vec{w}\rangle$

Define $f'(a) = \lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ (rate of change of at a)

Let $c_1,c_2\in\mathbb{R}$. Show that if $f'(a)$ and $g'(a)$ exist then

$$(c_1f+c_2g)'(a) = c_1f'(a) + c_2g'(a).$$

$$(c_1 f + c_2 g)'(a) = \lim_{h\to 0} \frac{(c_1 f + c_2 g)(a+h) - (c_1 f + c_2 g)(a)}{h}$$

$$= \lim_{h\to 0} \frac{c_1 f(a+h) + c_2 g(a+h) - (c_1 f(a) + c_2 g(a))}{h}$$

$$= \lim_{h\to 0} \frac{c_1 f(a+h) - c_1 f(a) + c_2 g(a+h) - c_2 g(a)}{h}$$

$$= \lim_{h\to 0} c_1 \left(\frac{f(a+h) - f(a)}{h}\right) + c_2 \left(\frac{g(a+h) - g(a)}{h}\right)$$

$$= c_1 \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + c_2 \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= c_1 f'(a) + c_2 g'(a).$$

__Problem__ Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. Show that $f'(a) = 1$.

__Solution__ $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{a+h-a}{h}$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} \frac{1}{1} = \lim_{h\to 0} 1 = 1. //$$

Problem Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$ and assume $f'(a)$ and $g'(a)$ exist.
Show that

$$(fg)'(a) = f(a)\,g'(a) + f'(a)\,g(a)$$

Solution

$$(fg)'(a) = \lim_{h\to 0} \frac{(fg)(a+h) - (fg)(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)\,g(a+h) - f(a)\,g(a)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(a+h)\,g(a+h) - f(a)\,g(a+h) + f(a)\,g(a+h) - f(a)\,g(a)}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{(f(a+h) - f(a))}{h}\, g(a+h) + f(a) \frac{(g(a+h) - g(a))}{h} \right)$$

$$= f'(a)\, g(a+0) + f(a)\, g'(a)$$

$$= f'(a)\, g(a) + f(a)\, g'(a).$$