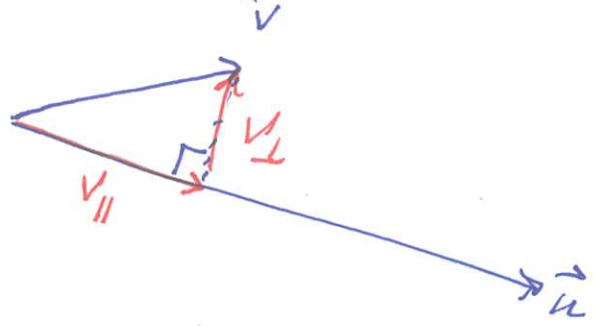
Problem Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Derive the formula for $\mathrm{proj}_{\vec{u}}(\vec{v})$.

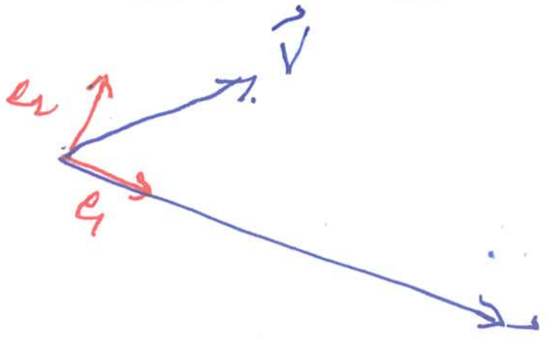
Solution:



$\vec{v} = v_\parallel + v_\perp$

$v_\parallel = \mathrm{proj}_{\vec{u}}(\vec{v})$.

Main idea: Perpendicular unit vectors are easier to work with



- $\vec{e}_1 = \frac{1}{\|\vec{u}\|}\vec{u}$, $\quad \langle \vec{e}_1, \vec{e}_2 \rangle = 0$, $\quad \langle \vec{e}_2, \vec{e}_2 \rangle = 1$
  $\langle \vec{e}_1, \vec{e}_1 \rangle = 1$.

There are $c_1, c_2 \in \mathbb{R}$ such that

$v_\parallel = c_1 e_1$ and $v_\perp = c_2 e_2$.

Can we find c_1 and c_2?

$\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2$ (since $v = v_\parallel + v_\perp$).

So $\langle \vec{v}, \vec{e}_1 \rangle = \langle c_1\vec{e}_1 + c_2\vec{e}_2, \vec{e}_1 \rangle$

$= c_1\langle \vec{e}_1, \vec{e}_1 \rangle + c_2\langle \vec{e}_2, \vec{e}_1 \rangle$

$= c_1 \cdot 1 + c_2 \cdot 0 = c_1$

So $c_1 = \langle \vec{v}, \vec{e}_1 \rangle = \langle \vec{v}, \frac{1}{\|\vec{u}\|}\vec{u} \rangle = \frac{1}{\|\vec{u}\|}\langle \vec{v}, \vec{u} \rangle$.

So $v_{\|} = c_1 e_1 = \dfrac{1}{\|\vec{u}\|}\langle \vec{v}, \vec{u}\rangle \cdot \dfrac{1}{\|\vec{u}\|}\vec{u}$

$$= \frac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}.$$

So $\operatorname{proj}_{\vec{u}}(v) = \dfrac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}.$

<u>Problem</u> Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Show that

$$\cos(\theta(\vec{u}, \vec{u})) = \frac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

<u>Solution</u>

$$\cos(\theta(\vec{u}, \vec{v})) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{\|\vec{v}_{\|}\|}{\|\vec{v}\|} = \frac{\dfrac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\|\vec{u}\|}{\|\vec{v}\|}$$

$$= \frac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\| \cdot \|\vec{v}\|} \quad //.$$

<u>Remark</u> Since $v_{\perp} = c_2 e_2$ then $\|\vec{v}_{\perp}\| = c_2$

and $c_2 = \|\vec{v}_{\perp}\| = \|\vec{v} - \vec{v}_{\|}\| = \left\|\vec{v} - \dfrac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}\right\|$

and $e_2 = \dfrac{\vec{v}_{\perp}}{\|\vec{v}_{\perp}\|} = \dfrac{1}{\|v_{\perp}\|}\left(\vec{v} - \dfrac{\langle \vec{v}, \vec{u}\rangle}{\|\vec{u}\|^2}\vec{u}\right).$

Problem Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ and assume that $f'(a)$ and $g'(a)$ exist. Show that

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

Solution

$$(fg)'(a) = \lim_{h \to 0} \frac{(fg)(a+h) - (fg)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\begin{array}{l} f(a+h)g(a+h) - f(a)g(a+h) \\ \qquad + f(a)g(a+h) - f(a)g(a) \end{array} \right)}{h}$$

$$= \lim_{h \to 0} \frac{(f(a+h) - f(a))}{h} g(a+h) + f(a) \frac{(g(a+h) - g(a))}{h}$$

$$= f'(a)g(a+0) + f(a)g'(a)$$

$$= f'(a)g(a) + f(a)g'(a). \quad //.$$

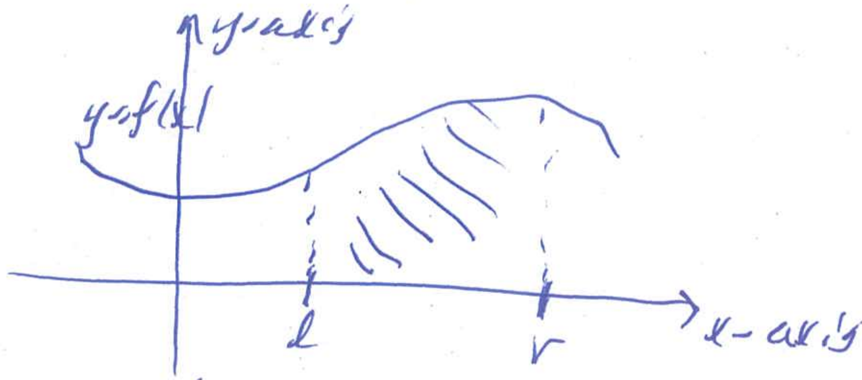
Fundamental theorem of Calculus

Suppose $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ and

$$A = \int f\,dx. \qquad \begin{pmatrix} A \text{ is the} \\ \text{integral of } f \end{pmatrix}$$

Then

$$\begin{pmatrix} \text{Area under} \\ y = f(x) \\ \text{from } x = l \text{ to } x = r \end{pmatrix} = A(r) - A(l)$$



Said another way:

Let $A(p) = \begin{pmatrix} \text{Area under } y = f(x) \\ \text{from } x = l \text{ to } x = p \end{pmatrix}$

Then $\quad A'(a) = f(a) \qquad \begin{pmatrix} f \text{ is the} \\ \text{derivative of } A \end{pmatrix}$

<u>Proof</u>

$$A'(a) = \lim_{h \to 0} \left(\frac{A(a+h) - A(a)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\begin{pmatrix} \text{Area under } y = f(x) \\ \text{from } x = a \text{ to } x = a+h \end{pmatrix}}{h} \right) = f(a)$$