21.05.2024 ①
Calculus Lect.
A. Ram

<u>Problem</u> Factor $1+z+z^2+\cdots+z^9$ in $\mathbb{C}[z]$ and in $\mathbb{R}[z]$.

$\frac{28}{7}=4$ BECAUSE $7\cdot 4=28$

$\frac{1-z^2}{1-z}=1+z$ BECAUSE $(1-z)(1+z)=1-z^2$

$\frac{1-z^3}{1-z}=1+z+z^2$ BECAUSE $(1-z)(1+z+z^2)=1+z+z^2$
$-z-z^2-z^3$
$=1-z^3$.

$\frac{1-z^{10}}{1-z}=1+z+z^2+\cdots+z^9$ BECAUSE

$(1-z)(1+z+\cdots+z^9)=1+z+\cdots+z^9$
$-z-\cdots-z^9-z^{10}$
$=1-z^{10}$.

<u>Problem</u> Prove by induction that
if $n\in\mathbb{Z}_{>0}$ then $\frac{1-z^n}{1-z}=1+z+\cdots+z^{n-1}$.

<u>Proof</u> <u>Base case</u> $n=1$. To show: $\frac{1-z^1}{1-z}=1$

This is true since $\frac{1}{1-z}$ is the thing such that
$\frac{1}{1-z}\cdot(1-z)=1$.

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Calculus sheet.

A. Ram

<u>Induction step:</u>

Assume $\dfrac{1-z^n}{1-z} = 1+z+\cdots+z^{n-1}.$

To show: $\dfrac{1-z^{n+1}}{1-z} = 1+z+\cdots+z^{n-1}+z^n.$

$$\frac{1-z^{n+1}}{1-z} = \frac{1-z^n+z^n-z^{n+1}}{1-z} = \frac{1-z^n}{1-u} + \frac{z^n-z^{n+1}}{1-z}$$

$$= \frac{1-z^n}{1-u} + z^n\frac{(1-z)}{1-z}$$

$$= 1+z+\cdots+z^{n-1}+z^n.$$

where the last line is by the induction assumption //.

<u>Problem</u> Prove, without using induction that if $n \in \mathbb{Z}_{>0}$ then $\dfrac{1-z^n}{1-z} = 1+z+\cdots+z^{n-1}.$

<u>Proof</u>

$$\frac{1-z^n}{1-z} = 1+z+\cdots+z^{n-1} \quad \text{BECAUSE}$$

$$\begin{aligned}(1-z)(1+z+\cdots+z^{n-1}) &= 1+z+\cdots+z^{n-1} \\ &\quad -z-z^2-\cdots-z^{n-1}-z^n \\ &= 1-z^n. \quad //\end{aligned}$$

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Calculus Lect.
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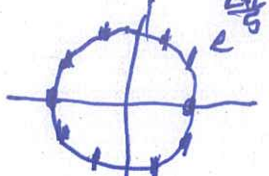
Problem Factor $1-z^{10}$ in $\mathbb{C}[z]$ and in $\mathbb{R}[z]$.

Solution: Since

$$\{z\in\mathbb{C} \mid z^{10}-1=0\} = \{z\in\mathbb{C} \mid z^{10}=1\}$$

$$= \left\{\begin{array}{l}\text{10 complex numbers on the unit} \\ \text{circle, equally spaced} \\ \text{including } 1\end{array}\right\}$$

$$= \Big\{$$  $$e^{\frac{\pi}{5}i} \Big\} = \left\{\begin{array}{l} e^{i\frac{4\pi}{5}},\ e^{i3\pi/5},\ e^{i2\pi/5},\ e^{i\pi/5}, \\ -1,\ e^{i\frac{6\pi}{5}},\ e^{i\frac{7\pi}{5}},\ e^{i\frac{8\pi}{5}},\ e^{i\frac{9\pi}{5}},\ 1,\end{array}\right\}$$

then

$$z^{10}-1 = (z-(-1))\cdot(z-e^{i\frac{4\pi}{5}})(z-e^{i\frac{3\pi}{5}})(z-e^{i\frac{2\pi}{5}})(z-e^{i\frac{\pi}{5}}) \cdot(z-e^{i\frac{6\pi}{5}})(z-e^{i\frac{7\pi}{5}})\cdot(z-e^{i\frac{8\pi}{5}})(z-e^{i\frac{9\pi}{5}})\cdot(z-1)$$

and

$$1+z+\cdots+z^9 = \frac{z^{10}-1}{z-1} = (z+1)\cdot(z-e^{i\frac{4\pi}{5}})(z-e^{i3\pi/5})(z-e^{i2\pi/5})(z-e^{i\pi/5}) \cdot(z-e^{-i\frac{4\pi}{5}})(z-e^{-i\frac{3\pi}{5}})(z-e^{-i\frac{2\pi}{5}})(z-e^{-i\pi/5})$$

To get the factorization in real numbers combine complex conjugate factors.

$$\overline{e^{i\pi/5}} = e^{-i\pi/5} \quad \text{and}$$

$$(z-e^{i\pi/5})(z-e^{-i\pi/5}) = z^2 - e^{i\pi/5}z - e^{-i\pi/5}z + 1$$

$$= z^2 - (e^{i\pi/5}+e^{-i\pi/5})z + 1 = z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1.$$

So

$$1+z+\cdots+z^9 = (z+1)\left(z^2-2\cos\left(\frac{\pi}{5}\right)z+1\right)\left(z^2-2\cos\left(\frac{2\pi}{5}\right)+1\right) \cdot\left(z^2-2\cos\left(\frac{3\pi}{5}\right)z+1\right)\left(z^2-2\cos\left(\frac{4\pi}{5}\right)z+1\right).$$

Problem Find horizontal and vertical tangent A Ram lines of $x^2+xy+y^2=9$.

Take the derivative of both sides.

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

So $(x+2y)\frac{dy}{dx}=-y-2x$.

So $$\frac{dy}{dx}=\frac{-y-2x}{x+2y}$$

Case 1: If $-y-2x=0$ then $\frac{dy}{dx}=0$ and we should get a slope 0 tangent line.

If $-y-2x=0$ and $x^2+xy+y^2=9$

then $y=-2x$ and $x^2+x(-2x)+(-2x)^2=9$.

So $x^2 - 2x^2+4x^2=9$. So $3x^2=9$ and $x^2=3$.

So $x=\sqrt{3}$ or $x=-\sqrt{3}$.

If $x=\sqrt{3}$ then $y=-2\sqrt{3}$

If $x=-\sqrt{3}$ then $y=2\sqrt{3}$.

(horizontal tangent lines at these points)

Case 2 If $x+2y=0$ then $\frac{dy}{dx}$ is probably infinite and we should get a slope ∞ (vertical) tangent line.

If $x + 2y = 0$ and $x^2 + xy + y^2 = 9$

then

$$y = \frac{-1}{2}x \quad \text{and} \quad x^2 + x\left(\frac{-1}{2}x\right) + \left(\frac{-1}{2}x\right)^2 = 9$$

and $x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 9$ so that $\frac{3}{4}x^2 = 9$.

So $x^2 = \frac{9 \cdot 4}{3} = 12$ so $x = \sqrt{12} = 2\sqrt{3}$ or $x = -\sqrt{12} = -2\sqrt{3}$

If $x = 2\sqrt{3}$ then $y = \frac{-1}{2} \cdot 2\sqrt{3} = -\sqrt{3}$

If $x = -2\sqrt{3}$ then $y = \frac{-1}{2} \cdot (-2\sqrt{3}) = \sqrt{3}$.

Let $x = z + w$
$y = z - w$.
Then $z = \frac{1}{2}(x+y)$
$w = \frac{1}{2}(x-y)$.

$$\begin{aligned}
9 = x^2 + xy + y^2 &= (z+w)^2 + (z+w)(z-w) + (z-w)^2 \\
&= z^2 + 2zw + w^2 \\
&\quad + z^2 \qquad\quad - w^2 \\
&\quad + z^2 - 2zw + w^2 \\
&= 3z^2 + w^2.
\end{aligned}$$

So $9 = 3z^2 + w^2$. If $w = 0$ then $z^2 = 3$
If $z = 0$ then $w^2 = 1$.