Lecture 29: Singular value decomposition

Let $t, s \in \mathbb{Z}_{>0}$ and let E_{ij} be the $t \times s$ matrix with 1 in the (i,j) entry and 0 elsewhere.
Let $A \in M_{t\times s}(\mathbb{R})$. Find orthonormal eigenvectors $v_1, \ldots, v_s$ of A^tA with eigenvalues $\lambda_1, \ldots, \lambda_s$ and let

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_s \\ | & & | \end{pmatrix} \quad \text{and} \quad S = \sqrt{\lambda_1}E_{11} + \cdots + \sqrt{\lambda_s}E_{ss}.$$

$$\text{If } \lambda_i \neq 0 \text{ let } u_i = \frac{1}{\sqrt{\lambda_i}}Av_i.$$

Extend $u_1, \ldots, u_k$ to an orthonormal basis of $\mathbb{R}^t$ and let

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_t \\ | & & | \end{pmatrix}.$$

Then $U \in O_t(\mathbb{R})$ and $V \in O_s(\mathbb{R})$ and $S \in M_{t\times s}(\mathbb{R})$ is ‘diagonal’ and

$$A = USV^T.$$

Example IP20. If

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{then} \quad A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The columns v_1, v_2, v_3 of

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{are orthonormal eigenvectors of } A^T A$$

with eigenvalues $\lambda = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$. Let

$$u_1 = \frac{1}{\sqrt{2}} A v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sqrt{1}} A v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\{u_1, u_2\}$ is already an orthonormal basis of $\mathbb{R}^2$, Let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{pmatrix}$$

Then $U \in O_2(\mathbb{R})$, $V \in O_3(\mathbb{R})$ and $S \in M_{2\times 3}(\mathbb{R})$ and $A = USV^T$.

Example IP19. If

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{then} \quad A^T A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The columns v_1, v_2 of

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{are orthonormal eigenvectors of } A^T A$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$. Let

$$u_1 = \frac{1}{\sqrt{1}} A v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and let} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

so that $\{u_1, u_2\}$ is an orthonormal basis of $\mathbb{R}^2$, Let

$$U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then $U \in O_2(\mathbb{R})$, $V \in O_2(\mathbb{R})$ and $S \in M_{2\times 2}(\mathbb{R})$ and $A = USV^T$.