Lecture 37: Orthogonality and linear independence

Definition (Orthogonal and orthonormal sequences.)

Let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$.

Let $u, v \in V$. The vectors u and v are

$$\textit{orthogonal} \quad \text{if } \langle u, v \rangle = 0.$$

An *orthogonal sequence* is a sequence $(b_1, \ldots, b_k)$ of vectors in V such that

$$\text{if } i, j \in \{1, \ldots, k\} \text{ and } i \neq j \quad \text{then} \quad \langle b_i, b_j \rangle = 0.$$

An *orthonormal sequence* is an orthogonal sequence $(b_1, \ldots, b_k)$ such that

$$\text{if } i \in \{1, \ldots, k\} \text{ then } \langle b_i, b_i \rangle = 1.$$

An *ordered orthonormal basis of V* is an orthonormal sequence $(b_1, \ldots, b_k)$ in V such that B is a basis of V.

Theorem (Pythagorean Theorem)

Let V be a $\mathbb{C}$-vector space with an inner product $\langle , \rangle : V \times V \to \mathbb{C}$. Let $u, v \in V$. If $\langle u, v \rangle = 0$ then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. Assume $\langle u, v \rangle = 0$.
To show $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} + \langle v, v \rangle \\ &= \|u\|^2 + 0 + \overline{0} + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2. \end{aligned}$$

□

Proposition (Orthogonal sets are linearly independent)

Let V be a vector space with inner product $\langle , \rangle \colon V \times V \to \mathbb{C}$.
Let $B = \{b_1, \ldots, b_k\}$ be an orthogonal set in V.
Then B is linearly independent.

Proof. Assume B is an orthogonal set in V.
To show: B is linearly independent.
To show: If $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$

$$\text{then } c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Assume $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$.

To show: $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

To show: If $i \in \{1, \ldots, k\}$ then $c_i = 0$.

Assume $i \in \{1, \ldots, k\}$. To show: $c_i = 0$.

$$\begin{aligned}
0 &= \langle c_1 b_1 + \cdots + c_k b_k, b_i \rangle \\
&= c_1 \langle b_1, b_i \rangle + \cdots + c_{i-1} \langle b_{i-1}, b_i \rangle + c_i \langle b_i, b_i \rangle \\
&\qquad + c_{i+1} \langle b_{i+1}, b_i \rangle + \cdots + c_k \langle b_k, b_i \rangle \\
&= c_1 \cdot 0 + \cdots + c_{i-1} \cdot 0 + c_i \langle b_i, b_i \rangle \\
&\qquad + c_{i+1} \cdot 0 + \cdots + c_k \cdot 0 \\
&= c_i \langle b_i, b_i \rangle.
\end{aligned}$$

Since $\langle , \rangle$ is an inner product and $b_i \neq 0$ then $\langle b_i, b_i \rangle \neq 0$. So

$$c_i = \frac{1}{\langle b_i, b_i \rangle} \cdot 0 = 0.$$

So B is linearly independent. □

Lecture 38: Linear transformations

Linear transformations are for comparing vector spaces.

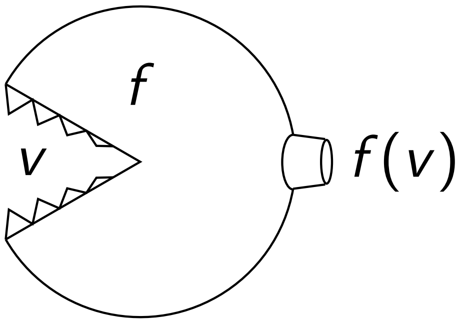
Definition (Linear transformation)

Let $\mathbb{F}$ be a field and let V and W be $\mathbb{F}$-vector spaces. An *$\mathbb{F}$-linear transformation from V to W* is a function $f\colon V \to W$ such that

(a) If $v_1, v_2 \in V$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,

(b) If $c \in \mathbb{F}$ and $v \in V$ then $f(cv) = cf(v)$.

V



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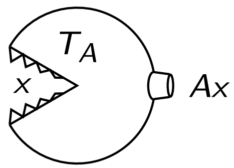
Example A2. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T_A\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T_A(x) = Ax.$$



Show that T_A is a linear transformation.
Let $u, v \in \mathbb{R}^s$. Then, by the distributive property of matrix multiplication for matrices,

$$T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v).$$

Let $u \in \mathbb{R}^s$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T_A(cu) = Acu = cAu = cT_A(u).$$

So T_A is a linear transformation.

Let $T\colon V \to W$ be a linear transformation. Assume that T has an inverse function $T^{-1}\colon W \to V$. Show that T^{-1} is a linear transformation.
Assume $w_1, w_2 \in W$. Then

$$\begin{aligned} T^{-1}(w_1 + w_2) &= T^{-1}(T(T^{-1}(w_1)) + T(T^{-1}(w_2))) \\ &= T^{-1}(T(T^{-1}(w_1) + T^{-1}(w_2)) \\ &= T^{-1}(w_1) + T^{-1}(w_2), \end{aligned}$$

where the first equality is because $T \circ T^{-1} = \mathrm{Id}$), the second equality is because T is a linear transformation) and the third equality is because $T^{-1} \circ T = \mathrm{Id}$. Assume $w \in W$ and $c \in \mathbb{R}$. Then

$$T^{-1}(cw) = T^{-1}(c \cdot T(T^{-1}(w))) = T^{-1}T(c \cdot T^{-1}(w)) = c \cdot T^{-1}(w).$$

So T^{-1} is a linear transformation.