05.02.2026 (1)

Linear Algebra A. Ram

Lect. 25.

Proof Machine

Three kinds of proof tasks:

(I) LHS = RHS.

(II) If __A__ then __B__

(III) There exists __C__ such that __D__

The proof machine simply doesn't work without definitions (in proper mathematical grammar).

Amazing theorems in Linear algebra

__Theorem__ (Pythagorean theorem)
Let V be vector space with an inner product $\langle\ \rangle : V \times V \to \mathbb{C}$. If $u, v \in V$ are orthogonal then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

__Theorem__

ker, im and span are subspaces.

<u>Theorem</u> If B is an orthogonal set in V then B is linearly independent.

<u>Theorem</u> If A is a matrix and $\ker(A) = \{0\}$ then the columns of A are linearly independent.

<u>Theorem</u> Any two bases of V have the same number of elements.

<u>Theorem</u> Invertible matrices are square.

<u>Theorem</u> (Pythagorean Theorem)
Assume V is a $\mathbb{C}$-vector space with $\langle\,,\rangle : V \times V \to \mathbb{C}$. Let $u, v \in V$.
If $\langle u, v\rangle = 0$ then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

<u>Proof</u> Assume $\langle u, v\rangle = 0$.
To show: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} + \langle v,v\rangle$$

$$= \|u\|^2 + 0 + \overline{0} + \|v\|^2$$

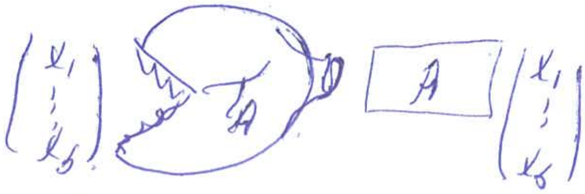
$$= \|u\|^2 + \|v\|^2. \quad //$$

<u>Matrices give linear transformations</u>

<u>Proposition</u> Let $s,t \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$

Define

$$T_A : \mathbb{R}^s \to \mathbb{R}^t$$
$$x \longmapsto Ax$$



Then T_A is a linear transformation.

<u>Proof</u> To show: (1) If $v_1, v_2 \in \mathbb{R}^s$ then

$$T_A(v_1+v_2) = T_A(v_1) + T_A(v_2)$$

(2) If $v \in \mathbb{R}^s$ and $c \in \mathbb{R}$ then

$$T_A(cv) = cT_A(v).$$

(1) Assume $v_1, v_2 \in \mathbb{R}^s$. To show:

$$T_A(v_1+v_2) = T_A(v_1) + T_A(v_2).$$

$$T_A(v_1+v_2) = A(v_1+v_2) = Av_1 + Av_2$$

$$= T_A(v_1) + T_A(v_2).$$

(2) Assume $v \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

To show: $T_A(cv) = cT_A(v)$.

$$T_A(cv) = A(cv) = cAv = cT_A(v).$$

So T_A is a linear transformation //.

<u>Span is a subspace</u>

<u>Proposition</u> Let V be a vector space. Let $B = \{b_1, \ldots, b_k\}$ be a subset of V. Then $\mathrm{span}(B)$ is a subspace of V.

<u>Proof</u>: To show:

(1) $0 \in \mathrm{span}(B)$

(2) If $v_1, v_2 \in \mathrm{span}(B)$ then $v_1 + v_2 \in \mathrm{span}(B)$

(3) If $v \in \mathrm{span}(B)$ and $c \in \mathbb{R}$ then $cv \in \mathrm{span}(B)$

(1) Since $0 = 0b_1 + \cdots + 0b_k \in \mathrm{span}\{b_1, \ldots, b_k\}$ then $0 \in \mathrm{span}(B)$.

(2) Assume $v_1, v_2 \in \mathrm{span}(B)$.

To show: $v_1 + v_2 \in \mathrm{span}(B)$.

Let $v_1 = a_1 b_1 + \cdots + a_k b_k$ and

$v_2 = d_1 b_1 + \cdots + d_k b_k$

Then

$$v_1+v_2 = a_1b_1+\cdots+a_kb_k$$
$$+d_1b_1+\cdots+d_kb_k$$
$$=(a_1+d_1)b_1+\cdots+(a_k+d_k)b_k$$

So $v_1+v_2 \in \mathrm{span}\{b_1,\ldots,b_k\}$. So $v_1+v_2\in \mathrm{span}(B)$.

(3) Assume $v\in\mathrm{span}(B)$ and $c\in\mathbb{R}$.

To show: $cv\in\mathrm{span}(B)$.

Since $v\in\mathrm{span}\{b_1,\ldots,b_k\}$ then there exist $d_1,\ldots,d_k\in\mathbb{R}$ such that

$$v = d_1b_1+\cdots+d_kb_k$$

Then

$$cv = c(d_1b_1+\cdots+d_kb_k)$$
$$=(cd_1)b_1+\cdots+(cd_k)b_k.$$

So $cv\in\mathrm{span}\{b_1,\ldots,b_k\}$. So $cv\in\mathrm{span}(B)$.

So $\mathrm{span}(B)$ is a subspace of V.