Lecture 39: Subspaces

Definition (Kernel and image of a linear transformation)

The *kernel* of an $\mathbb{F}$-linear transformation $f : V \to W$ is the set

$$\ker(f) = \{v \in V \mid f(v) = 0\}.$$

The *image* of an $\mathbb{F}$-linear transformation $f : V \to W$ is the set

$$\operatorname{im}(f) = \{f(v) \mid v \in V\}.$$

Definition (Kernel and image of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *kernel of A* is

$$\ker(A) = \{x \in \mathbb{Q}^s \mid Ax = 0\}$$

and the *image of A* is

$$\operatorname{im}(A) = \{Ax \mid s \in \mathbb{Q}^s\}.$$

A *subspace of* $\mathbb{Q}^s$ is a subset $W \subseteq \mathbb{Q}^s$ such that

(a) $0 \in W$,

(b) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,

(c) If $w \in W$ and $c \in \mathbb{Q}$ then $cw \in W$.

Proposition

Let $A \in M_{t\times s}(\mathbb{Q})$. Then $\ker(A)$ is a subspace of $\mathbb{Q}^s$.

Proof. (a) Since $A0 = 0$ then $0 \in \ker(A)$.

(b) Assume $w_1, w_2 \in \ker(A)$. Then $Aw_1 = 0$ and $Aw_2 = 0$. So

$$A(w_1 + w_2) = Aw_1 + Aw_2 = 0 + 0 = 0. \quad \text{So } w_1 + w_2 \in \ker(A).$$

(c) Assume $w \in \ker(A)$ and $c \in \mathbb{Q}$. Then $Aw = 0$ and

$$A(cw) = cAw = c0 = 0. \quad \text{So } cw \in \ker(A).$$

So $\ker(A)$ is a subspace of $\mathbb{Q}^s$. □

A *subspace* of $\mathbb{Q}^t$ is a subset $Y \subseteq \mathbb{Q}^t$ such that

(a) $0 \in Y$,

(b) If $y_1, y_2 \in Y$ then $y_1 + y_2 \in Y$,

(c) If $y \in Y$ and $c \in \mathbb{Q}$ then $cy \in Y$.

Proposition

Let $A \in M_{t\times s}(\mathbb{Q})$. Then $im(A)$ is a subspace of $\mathbb{Q}^t$.

Proof. (a) Since $0 = A0$ then $0 \in \mathrm{im}(A)$.

(b)Assume $y_1, y_2 \in \mathrm{im}(A)$. Then there exist $x_1, x_2 \in \mathbb{Q}^s$ such that $y_1 = Ax_1$ and $y_2 = Ax_2$. Then

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2). \quad \text{So } y_1 + y_2 \in \mathrm{im}(A).$$

(c) Assume $y \in \mathrm{im}(A)$ and $c \in \mathbb{Q}$. Then there exists $x \in \mathbb{Q}^s$ such that $y = Ax$. Then

$$cy = cAx = A(cx). \quad \text{So } cy \in \mathrm{im}(A).$$

So $\mathrm{im}(A)$ is a subspace of $\mathbb{Q}^t$. $\square$

Example A5. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation. Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

Let $v_1, v_2 \in \ker(T)$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0. \quad \text{So } v_1 + v_2 \in \ker(T).$$

Subtracting $T(0)$ from each side of the equation $T(0) = T(0+0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \ker(T).$$

Let $v \in \ker(T)$ and let $c \in \mathbb{R}$. Then

$$T(cv) = cT(v) = c \cdot 0 = 0 \quad \text{and so} \quad cv \in \ker(T).$$

So $\ker(T)$ is a subspace of V.

Example A6. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation. Show that $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.
Subtracting $T(0)$ from each side of the equation $T(0) = T(0+0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \mathrm{im}(T).$$

Let $w_1, w_2 \in W$. Then there exist $v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \quad \text{and} \quad T(v_2) = w_2.$$

Then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$,

$$\text{and so} \quad w_1 + w_2 \in \mathrm{im}(T).$$

Let $w \in W$ and let $c \in \mathbb{R}$. Then there exists $v \in V$ such that

$$T(v) = w.$$

Then $cw = cT(v) = T(cv)$

$$\text{and so} \quad cw \in \mathrm{im}(T).$$

So $\mathrm{im}(T)$ is a subspace of W.

Example V27&28. Let

$$S = \{|1, 3, -1, 1\rangle,\ |2, 6, 0, 4\rangle,\ |3, 9, -2, 4\rangle\ \}.$$

Then

$$S = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ -2 \\ 4 \end{pmatrix} \right\}$$

and

$$\mathbb{R}\text{-span}(S) = \mathrm{im}(A),\ \text{where } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

Lecture 40: The minimax basis theorem

Definition (Spanning set, linearly independent set, basis)

Let V be an $\mathbb{F}$-vector space and let $B = \{v_1, \ldots, v_k\}$ be a subset of V. The subset B is a *spanning set of V* if B satisfies

$$\{c_1 v_1 + \cdots + c_k v_k \mid c_1, \ldots, c_k \in \mathbb{F}\} = V.$$

The subset B is a *linearly independent set in V* if B satisfies

$$\text{if } c_1, \ldots, c_k \in \mathbb{F} \text{ and } c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0$$

$$\text{then } c_1 = 0, \ldots, c_k = 0.$$

The subset B is a *basis of V* if B satisfies:

B is a spanning set of V and B is a linearly independent set in V.

Theorem (Basis Minimax Theorem)

Let V be an $\mathbb{F}$-vector space and let B be a subset of V. The following are equivalent:

(a) *B is a basis of V.*

(b) *B is a minimal spanning set of V.*

(c) *B is a maximal linearly independent set of V.*

Theorem (Exchange Theorem)

Let V be an $\mathbb{F}$-vector space. Let $B = \{v_1, \ldots, v_k\}$ be a basis of V and let $D = \{d_1, \ldots, d_\ell\}$ be another basis of V. Then there exists $d_{i_1} \in D$ such that

$$\{d_{i_1}, b_2, b_3, \ldots, b_k\} \quad \text{is a basis of } V.$$

Theorem (Dimension Theorem)

Let V be an $\mathbb{F}$-vector space. Any two bases of V have the same number of elements.

The Dimension Theorem is the reason that

$$\dim(V) \text{ makes sense to consider.}$$

Definition (Dimension)

Let V be a vector space. The *dimension* of V is

$$\dim(V) = (\text{number of elements in a basis } B \text{ of } V).$$

The following provides an example of a spanning set that is not minimal, and another spanning set for the same subspace that is minimal.

Example V14. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1),(2,2,2),(3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

In this case

$$\begin{aligned}
\mathbb{R}\text{-span}(S) &= \{c_1\,|1,1,1\rangle + c_2\,|2,2,2\rangle + c_3\,|3,3,3\rangle \mid c_1,c_2,c_3 \in \mathbb{R}\} \\
&= \{c_1\,|1,1,1\rangle + 2c_2\,|1,1,1\rangle + 3c_3\,|1,1,1\rangle \mid c_1,c_2,c_3 \in \mathbb{R}\} \\
&= \{(c_1+2c_2+3c_3)\,|1,1,1\rangle \mid c_1,c_2,c_3 \in \mathbb{R}\} \\
&= \{t\,|1,1,1\rangle \mid t \in \mathbb{R}\} = \mathbb{R}\text{-span}\{|1,1,1\rangle\} \\
&= \{|t,t,t\rangle \mid t \in \mathbb{R}\}.
\end{aligned}$$

Here $\{\ |1,1,1\rangle\ \}$ is a basis of $\mathbb{R}$-span(S) and

$$\dim(\mathbb{R}\text{-span}(S)) = 1 \quad \text{(even though } S \text{ has 3 elements).}$$

□

Proposition (Span is a subspace)

Let V be a vector space. Let $B = \{b_1, \ldots, b_k\}$ be a subset of V. Then span(B) is a subspace of V.

Proof.

To show: (1) $0 \in \text{span}(B)$.
(2) If $v_1, v_2 \in \text{span}(B)$ then $v_1 + v_2 \in \text{span}(B)$.
(3) If $v \in \text{span}(B)$ and $c \in \mathbb{R}$ then $cv \in \text{span}(B)$.

(1) Since $0 = 0b_1 + \cdots 0b_k$ then $0 \in \text{span}\{b_1, \ldots, b_k\} = \text{span}(B)$.

(2) Assume $v_1, v_2 \in \text{span}(B)$. To show $v_1 + v_2 \in \text{span}(B)$.
Since $v_1, v_2 \in \text{span}(B)$
then there exist $a_1, \ldots, a_k, c_1, \ldots, c_k \in \mathbb{R}$ such that

$$v_1 = a_1 b_1 + \cdots + a_k b_k \quad \text{and} \quad v_2 = c_1 b_1 + \cdots + c_k b_k.$$

Then

$$\begin{aligned} v_1 + v_2 &= (a_1 b_1 + \cdots + a_k b_k) + (c_1 b_1 + \cdots + c_k b_k) \\ &= (a_1 + c_1) b_1 + \cdots + (a_k + c_k) b_k. \end{aligned}$$

So $v_1 + v_2 \in \text{span}\{b_1, \ldots, b_k\} = \text{span}(B)$.

(3) Assume $v \in \text{span}(B)$ and $c \in \mathbb{R}$.
To show $cv \in \text{span}(B)$.
Since $v \in \text{span}(B)$ then there exist $a_1, \ldots, a_k \in \mathbb{R}$ such that

$$v = a_1 b_1 + \cdots + a_k b_k.$$

Then

$$cv = c(a_1 b_1 + \cdots + a_k b_k) = (ca_1) b_1 + \cdots + (ca_k) b_k.$$

So $cv \in \text{span}\{b_1, \ldots, b_k\}$. So $cv \in \text{span}(B)$. $\square$

Proof of the Dimension Theorem

Assume

$$B = \{b_1, \ldots, b_k\} \quad \text{is a basis of } V \text{ and}$$
$$D = \{d_1, \ldots, d_\ell\} \quad \text{is another basis of } V.$$

Using the Exchange theorem, there exists $d_{i_1} \in D$ such that $d_{i_1} \notin \operatorname{span}(B - b_1)$. Then

$$B_1 = \{d_{i_1}, b_2, b_3, b_4, \ldots b_k\} \quad \text{is a basis of } V.$$

Using the Exchange theorem, there exists $d_{i_2} \in D$ such that $d_{i_2} \notin \operatorname{span}(B_1 - b_2)$. Then

$$B_2 = \{d_{i_1}, d_{i_2}, b_3, b_4, \ldots b_k\} \quad \text{is a basis of } V.$$

Continue this replacement process to obtain

$$B' = \{d_{i_1}, \ldots, d_{i_k}\} \subseteq D, \quad \text{such that } B' \text{ is a basis of } V.$$

By the Minimax Theorem D is a minimal spanning set.
So $B' = D$ and $k = \ell$. □

Proof of the Exchange Theorem

Assume

$$B = \{b_1, \dots, b_k\} \quad \text{is a basis of } V \text{ and}$$
$$D = \{d_1, \dots, d_\ell\} \quad \text{is another basis of } V.$$

If $d_1, \dots, d_\ell \in \text{span}(B - \{b_1\})$ then

$$V = \text{span}(d_1, \dots, d_\ell) \subseteq \text{span}(B - \{b_1\}) \subseteq V$$

giving $V = \text{span}(B - \{b_1\})$.

But since B is a minimal spanning set then $V \neq \text{span}(B - \{b_1\})$ and so

$$\text{there exists } d_{i_1} \in D \text{ such that } d_{i_1} \notin \text{span}(B - \{b_1\}).$$

$$d_{i_1} = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k, \quad \text{with } c_1 \neq 0.$$

To show: $B_1 = \{d_{i_1}, b_2, \dots, b_k\}$ is a basis of V.

To show: (1) $\text{span}\{d_{i_1}, b_2, \dots, b_k\} = V$.
(2) $\{d_{i_1}, b_2, \dots, b_k\}$ is linearly independent.

(1) Since

$$b_1 = c_1^{-1}(-d_{i_1} + c_2 b_2 + \cdots + c_k v_k)$$

then $b_1, b_2, \dots, b_k \in \text{span}\{d_{i_1}, b_2, \dots, b_k\}$. So
$V = \text{span}\{b_1, \dots, b_k\} \subseteq \text{span}\{d_{i_1}, b_2, \dots, b_k\} \subseteq V$. So

$$V = \text{span}(d_{i_1}, b_2, \dots, b_k\}.$$

(2) If $a_1 d_{i_1} + a_2 b_2 + \cdots + a_k b_k = 0$ then

$$a_1(c_1 b_1 + c_2 b_2 + \cdots + c_k b_k) + a_2 b_2 + \cdots + a_k b_k = 0.$$

Since B is linearly independent then $a_1 c_1 = 0$.
Since $c_1 \neq 0$ then $a_1 = 0$ and $a_2 b_2 + \cdots + a_k b_k = 0$.
Since B is linearly independent then $a_2 = 0, \dots, a_k = 0$.
So $\{d_{i_1}, b_2, \dots, b_k\}$ is linearly independent.
So $\{d_{i_1}, b_2, \dots, b_k\}$ is a basis of V. □

Proof of the Minimax Basis Theorem

(a) $\Rightarrow$ (b): Assume $B = \{b_1, \ldots, b_k\}$ is a basis of B.
To show: B is a minimal spanning set of V.
To show: (1) B Is a spanning set.
(2) If $i \in \{1, \ldots, k\}$ then $B - \{b_i\}$ is not a spanning set.

(1) Since B is a basis then B is a spanning set.

(2) To show: If $i \in \{1, \ldots, k\}$ and $B - \{b_i\}$ is a spanning set then B is not a basis.
Assume $i \in \{1, \ldots, k\}$ and $B - \{b_i\}$ is a spanning set.
Then there exist $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in \mathbb{R}$ such that

$$b_i = c_1 b_1 + \cdots + c_{i-1} b_{i-1} + c_{i+1} b_{i+1} + \cdots + c_k b_k.$$

Then

$$0 = c_1 b_1 + \cdots + c_{i-1} b_{i-1} - b_i + c_{i+1} b_{i+1} + \cdots + c_k b_k.$$

So $\{b_1, \ldots, b_k\}$ is not linearly independent.
So B is not a basis.
So if $i \in \{1, \ldots, k\}$ then $B - \{b_i\}$ is not a spanning set.

(b) $\Rightarrow$ (c): Assume B is a minimal spanning set.
To show: B is a maximal linearly independent set in V.
To show: (1) B is a linearly independent set in V.
(2) If $v \in V$ then $B \cup \{v\}$ is not linearly independent.

(1) To show: If B is a spanning set and B is not linearly independent then B is not a minimal spanning set.
Assume B is a spanning set and B is not linearly independent.
Then there exist $c_1, \ldots, c_k \in \mathbb{R}$ and $i \in \{1, \ldots, k\}$ such that

$$c_1 b_1 + \cdots + c_k b_k = 0 \quad \text{and} \quad c_i \neq 0.$$

Then $b_i = -c_i^{-1}(c_1 b_1 + \cdots + c_{i-1} b_{i-1} + c_{i+1} b_{i+1} + \cdots + c_k b_k)$.
So $\text{span}(B - \{b_i\}) \supseteq \text{span}(b_1, \ldots, b_k) = V$.
So $\text{span}(B - \{b_i\}) = V$ and B is not a minimal spanning set of B.
So if B is a minimal spanning set then B is linearly independent.

(2) To show: If $v \in V$ then $B \cup \{v\}$ is not linearly independent.
Assume $v \in V$. To show: $B \cup \{v\}$ is not linearly independent.

Since $\mathrm{span}(B) = V$ then there exist $c_1, \ldots, c_k \in \mathbb{R}$ such that

$$v = c_1 b_1 + \cdots c_k b_k.$$

So $0 = c_1 b_1 + \cdots + c_k b_k - v$.
So $B \cup \{v\} = \{b_1, \ldots, b_k, v\}$ is not linearly independent.

(c) $\Rightarrow$ (a): Assume B is a maximal linearly independent set.
To show: B is a basis.
To show: $\mathrm{span}(B) = V$.
Assume $v \in V$. To show $v \in \mathrm{span}(B)$.
Since B is a maximal linearly independent set then $B \cup \{v\}$ is not linearly independent.
So there exist $c_1, \ldots, c_k, c_{k+1} \in \mathbb{R}$ and $i \in \{1, \ldots, k+1\}$ such that

$$c_1 b_1 + \cdots + c_k b_k + c_{k+1} v = 0 \quad \text{and} \quad c_i \neq 0.$$

The case $c_{k+1} = 0$ cannot occur since B is linearly independent.
So $c_{k+1} \neq 0$ and $v = -c_{k+1}^{-1}(c_1 b_1 + \cdots c_k b_k)$.
So $v \in \mathrm{span}\{b_1, \ldots, b_k\} = \mathrm{span}(B)$.
So $V = \mathrm{span}(B)$. So B is a basis of V. $\square$