Bases

Theorem (The dimension theorem)
Let V be a vector space.
Let $B = \{b_1, \dots, b_k\}$ be a basis of V.
Let $D = \{d_1, \dots, d_\ell\}$ be another basis of V.
Then
$$k = \ell.$$

The proof of this theorem uses two others.

Theorem (The exchange theorem)
Let $B = \{b_1, \dots, b_k\}$ be a basis of V.
Let $D = \{d_1, \dots, d_\ell\}$ be another basis of V.
Then there exists $d_{i_1} \in D$ such that
$$\{d_{i_1}, b_2, \dots, b_k\} \text{ is a basis of } V.$$

Theorem (The minimax theorem)
The following are equivalent:
(a) B is a basis of V
(b) B is a minimal spanning set of V.
(c) B is a maximal linearly independent set in V.

Let V be a vector space.

A <u>spanning set of V</u> is a subset $B=\{b_1,\ldots,b_k\}$ of V such that

$$V=\mathrm{span}\{b_1,\ldots,b_k\}.$$

A <u>linearly independent subset of V</u> is a subset $B=\{b_1,\ldots,b_k\}$ of V such that

if $c_1,\ldots,c_k\in\mathbb{R}$ and $c_1b_1+\cdots+c_kb_k=0$
then $c_1=0$, $c_2=0$, $\ldots$, $c_k=0$.

A <u>basis of V</u> is a subset $B=\{b_1,\ldots,b_k\}$ of V such that

B is a spanning set of V
<u>and</u> B is a linearly independent set of V.

<u>Proof of the dimension theorem</u>

Let $B=\{b_1,\ldots,b_k\}$ be a basis of V
Let $D=\{d_1,\ldots,d_\ell\}$ be another basis of V.
Use the <u>exchange theorem</u> to make a basis

$$B_1=\{d_1,b_2,\ldots,b_k\} \text{ of } V.$$

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Use the <u>exchange theorem</u> again to make a basis

$$B_2 = \{d_{i_1}, d_{i_2}, b_3, \ldots, b_k\} \text{ of } V.$$

Continue this process to make a basis

$$B_k = \{d_{i_1}, d_{i_2}, \ldots, d_{i_k}\} \text{ of } V.$$

Then $B_k \subseteq D$. By the <u>minimax theorem</u> D is a minimal spanning set of V. So $B_k = D$. So $k = \ell$. //

<u>Proof of the Exchange theorem</u>

Let $B = \{b_1, \ldots, b_k\}$ be a basis of V.
Let $D = \{d_1, \ldots, d_\ell\}$ be another basis of V.

If $d_1, \ldots, d_\ell \in \mathrm{span}(B - \{b_1\})$ then
$V = \mathrm{span}(d_1, \ldots, d_\ell) \subseteq \mathrm{span}(B - \{b_1\})$ and
$V = \mathrm{span}(B - \{b_1\})$.

By the <u>minimax theorem</u> $V \neq \mathrm{span}(B - \{b_1\})$.
So there exists d_{i_1} such that

$$d_{i_1} \notin \mathrm{span}(B - \{b_1\})$$

To show: $\{d_{i_1}, b_2, \dots, b_k\}$ is a basis of V.

To show: (1) $\mathrm{span}\{d_{i_1}, b_2, \dots, b_k\} = V$

(2) $\{d_{i_1}, b_2, \dots, b_k\}$ is linearly independent

We know there exist $c_1, \dots, c_k \in \mathbb{R}$ such that

$$d_{i_1} = c_1 b_1 + \dots + c_k b_k \quad \text{with } c_1 \neq 0.$$

So $b_1 = c_1^{-1}(d_{i_1} - c_2 b_2 - \dots - c_k b_k)$.

(1) So $b_1 \in \mathrm{span}\{d_{i_1}, b_2, \dots, b_k\}$.

So $\mathrm{span}\{b_1, b_2, \dots, b_k\} \subseteq \mathrm{span}\{d_{i_1}, b_2, \dots, b_k\}$.

So $V \subseteq \mathrm{span}\{d_{i_1}, b_2, \dots, b_k\} \subseteq V$.

So $V = \mathrm{span}\{d_{i_1}, b_2, \dots, b_k\}$.

(2) To show: If $a_1, \dots, a_k \in \mathbb{R}$ and

$a_1 d_{i_1} + a_2 b_2 + \dots + a_k b_k = 0$ then $a_1 = 0, a_2 = 0, \dots, a_k = 0$.

Assume $a_1, \dots, a_k \in \mathbb{R}$ and $a_1 d_{i_1} + a_2 b_2 + \dots + a_k b_k = 0$.

then $a_1(c_1 b_1 + \dots + c_k b_k) + a_2 b_2 + \dots + a_k b_k = 0$

Since $\{b_1, \dots, b_k\}$ is linearly independent

then $a_1 c_1 = 0$.

Since $c_1 \neq 0$ then $a_1 = 0$.

$\therefore$

$a_2 b_2 + \cdots + a_k b_k = 0.$

Since $\{b_1, b_2, \ldots, b_k\}$ is linearly independent then $a_2 = 0, a_3 = 0, \ldots, a_k = 0.$

$\therefore a_1 = 0, a_2 = 0, \ldots, a_k = 0.$

$\therefore \{d_{i_1}, b_2, \ldots, b_k\}$ is linearly independent.

$\therefore \{d_{i_1}, b_2, \ldots, b_k\}$ is a basis of V. //