Lecture 41: Kernel, image, linear independence and span

Definition (Kernel and image of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *kernel of A* is

$$\ker(A) = \{x \in \mathbb{Q}^s \mid Ax = 0\}$$

and the *image of A* is

$$\operatorname{im}(A) = \{Ax \mid s \in \mathbb{Q}^s\}.$$

Proposition

Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.
If $\ker(A) = 0$ then the columns of A are linearly independent.

Proof. Let $a_1, \ldots, a_s$ be the columns of A.
Assume $\ker(A) = 0$.
To show: $a_1, \ldots, a_s$ are linearly independent.
To show: If $c_1, \ldots, c_s \in \mathbb{R}$ and $c_1 a_1 + \cdots + c_s a_s = 0$

$$\text{then } c_1 = 0, c_2 = 0, \ldots, c_s = 0.$$

Assume $c_1, \ldots, c_s \in \mathbb{R}$ and $c_1 a_1 + \cdots + c_s a_s = 0$.
Then

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix} = 0. \qquad \text{So } \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix} \in \ker(A). \qquad \text{So } \begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So $c_1 = 0, c_2 = 0, \ldots, c_s = 0$.
So $\{a_1, \ldots, a_s\}$ is linearly independent. □

Proposition

Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Let $a_1, \dots, a_s$ be the columns of A. Then

$$im(A) = span\{a_1, \dots, a_s\}.$$

Proof.

$$\begin{aligned}
\mathrm{im}(A) = \{Ax \mid x \in \mathbb{R}^s\} &= \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_s \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \;\middle|\; x_1, \dots, x_s \in \mathbb{R} \right\} \\
&= \left\{ x_1 \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + x_s \begin{pmatrix} | \\ a_s \\ | \end{pmatrix} \;\middle|\; x_1, \dots, x_s \in \mathbb{R} \right\} \\
&= \mathbb{R}\text{-span}\{\text{columns of } A\}.
\end{aligned}$$

So $\mathrm{im}(A)$ is the set of linear combinations of the columns of A. □

Lecture 42: Invertible matrices are square

Theorem (Invertible matrices are square)

Let $s, t \in \mathbb{Z}_{>0}$ *and let* $A \in M_{t\times s}(\mathbb{R})$*. Suppose there exists*

$$P \in M_{s\times t}(\mathbb{R}) \quad \textit{be such that} \quad PA = 1.$$

Suppose there exists

$$Q \in M_{s\times t}(\mathbb{R}) \quad \textit{be such that} \quad AQ = 1.$$

Then

(a) $\ker(A) = 0$.

(b) $im(A) = \mathbb{R}^t$.

(c) *The set of columns of* A *is a basis of* $\mathbb{Q}^t$.

(d) $s = t$.

(e) $P = Q$.

Proof. (a) To show: $\ker(A) = \{0\}$.

To show: (1) $\{0\} \subseteq \ker(A)$.
(2) $\ker(A) \subseteq \{0\}$.

(1) Since $A \cdot 0 = 0$ then $0 \in \ker(A)$.
So $\{0\} \subseteq \ker(A)$.

(2) To show: If $x \in \ker(A)$ then $x \in \{0\}$.
Assume $x \in \ker(A)$. To show: $x = 0$.
Since $x \in \ker(A)$ then

$$Ax = 0. \quad \text{So } PAx = P0 = 0.$$

So $x = 1x = PAx = 0$. So $\ker(A) \subseteq \{0\}$.

So $\ker(A) = \{0\}$.

(b) To show: $\text{im}(A) = \mathbb{R}^t$.

To show: (1) $\text{im}(A) \subseteq \mathbb{R}^t$.
(2) $\mathbb{R}^t \subseteq \text{im}(A)$.

(1) By definition of $(\text{im}(A) = \{Ax \mid x \in \mathbb{R}^s\}$.
Since A is a $t \times s$ matrix then $\text{im}(A) \subseteq \mathbb{R}^t$.

(2) To show: If $v \in \mathbb{R}^t$ then $v \in \text{im}(A)$.
Assume $v \in \mathbb{R}^t$. To show: $v \in \text{im}(A)$.

$$v = 1v = AQv \in \{Ax \mid x \in \mathbb{R}^t\} = \text{im}(A).$$

So $v \in \text{im}(A)$. So $\mathbb{R}^t \subseteq \text{im}(A)$.

So $\mathbb{R}^t = \text{im}(A)$.