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Linear Algebra (1)
Lect. 27 A. Ram

<u>Proposition</u> Let V be a vector space with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$.
Let $B = \{b_1, b_2, \ldots, b_k\}$ be an orthogonal set in V.
Then B is linearly independent.

<u>Proof</u> Assume B is an orthogonal set in V.
To show: B is linearly independent.
To show: If $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$
then $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.
Assume $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$.
To show: If $i \in \{1, \ldots, k\}$ then $c_i = 0$.
Assume $i \in \{1, \ldots, k\}$. To show: $c_i \neq 0$.

$$\begin{aligned}
0 &= \langle c_1 b_1 + \cdots + c_k b_k, b_i \rangle \\
&= c_1 \langle b_1, b_i \rangle + \cdots + c_{i-1} \langle b_{i-1}, b_i \rangle + c_i \langle b_i, b_i \rangle \\
&\qquad + c_{i+1} \langle b_{i+1}, b_i \rangle + \cdots + c_k \langle b_k, b_i \rangle \\
&= c_1 \cdot 0 + \cdots + c_{i-1} \cdot 0 + c_i \langle b_i, b_i \rangle \\
&\qquad + c_{i+1} \cdot 0 + \cdots + c_k \cdot 0 \\
&= c_i \langle b_i, b_i \rangle.
\end{aligned}$$

Since $\langle\ ,\ \rangle$ is an inner product and $b_i \neq 0$
then $\langle b_i, b_i \rangle \neq 0$. So
$c_i = \frac{1}{\langle b_i, b_i \rangle} \cdot 0 = 0$. So B is linearly indep. //

Proposition Let $A \in M_{t\times s}(\mathbb{R})$.
If $\ker(A)=0$ then the columns of A are linearly independent.

Proof Let $a_1, \ldots, a_s$ be the columns of A.
Assume $\ker(A)=0$.
To show: $a_1, \ldots, a_s$ are linearly independent.
To show: If $c_1, \ldots, c_s \in \mathbb{R}$ and $c_1a_1+\cdots+c_sa_s=0$
then $c_1=0, c_2=0, \ldots, c_s=0$.
Assume $c_1, \ldots, c_s \in \mathbb{R}$ and $c_1a_1+\cdots+c_sa_s=0$.
Then

$$A\begin{pmatrix} c_1 \\ \vdots \\ c_s\end{pmatrix}=0. \quad \text{So } \begin{pmatrix} c_1 \\ \vdots \\ c_s\end{pmatrix} \in \ker(A).$$

So

$$\begin{pmatrix} c_1 \\ \vdots \\ c_s\end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0\end{pmatrix}. \quad \text{So } c_1=0, \ldots, c_s=0.$$

So $\{a_1, \ldots, a_s\}$ is linearly independent. //

Proposition Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.
Then

$$\mathrm{im}(A) = \mathrm{span}\{\text{columns of } A\}.$$

Proof Assume $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$.

Let $a_1, a_2, \ldots, a_s$ be the columns of A.
Then

$$\operatorname{im}(A) = \left\{ A\begin{pmatrix} c_1 \\ \vdots \\ c_s \end{pmatrix} \;\middle|\; c_1, \ldots, c_s \in \mathbb{R} \right\}$$

$$= \{ c_1 a_1 + \cdots + c_s a_s \mid c_1, \ldots, c_s \in \mathbb{R} \}$$

$$= \operatorname{span}\{a_1, \ldots, a_s\}$$

$$= \operatorname{span}\{\text{columns of } A\}. \;//$$

Invertible matrices are square

Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.
Assume there exists $P \in M_{s\times t}(\mathbb{R})$ such that

$$PA = 1$$

Assume there exists $Q \in M_{s\times t}(\mathbb{R})$ such that

$$AQ = 1.$$

(a) $\ker(A) = 0$

(b) $\operatorname{im}(A) = \mathbb{R}^t$

(c) $\{\text{columns of } A\}$ is a basis of $\mathbb{R}^t$.

(d) $s = t$

(e) $P = Q$.

Proof (a) To show: $\ker(A) = \{0\}$.

To show: (1) $\{0\} \subseteq \ker(A)$

(2) $\ker(A) \subseteq \{0\}$.

(1) Since $A \cdot 0 = A \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ then $0 \in \ker(A)$.

So $\{0\} \subseteq \ker(A)$.

(2) To show: If $x \in \ker(A)$ then $x \in \{0\}$.

Assume $x \in \ker(A)$. To show: $x = 0$.

Since $x \in \ker(A)$ then

$$Ax = 0. \quad \text{So} \quad PAx = P \cdot 0 = 0.$$

So $x = 1 \cdot x = PAx = 0$. So $\ker(A) \subseteq \{0\}$

So $\ker(A) = \{0\}$.

(b) To show: $\operatorname{im}(A) = \mathbb{Q}^t$.

To show: (1) $\operatorname{im}(A) \subseteq \mathbb{Q}^t$.

(2) $\mathbb{Q}^t \subseteq \operatorname{im}(A)$.

(1) By definition of $\operatorname{im}(A) = \{Ax \mid x \in \mathbb{Q}^s\}$.

Since A is a $t \times s$ matrix then $\operatorname{im}(A) \subseteq \mathbb{Q}^t$.

(2) To show: If $v \in \mathbb{Q}^t$ then $v \in \operatorname{im}(A)$.

Assume $v \in \mathbb{Q}^t$. To show $v \in \operatorname{im}(A)$.

$v = 1 \cdot v = AQv \in \{A\underline{x} \mid \underline{x} \in \mathbb{Q}^s\} = \mathrm{im}(A)$.
So $v \in \mathrm{im}(A)$. So $\mathbb{Q}^t \subseteq \mathrm{im}(A)$.
So $\mathbb{Q}^t = \mathrm{im}(A)$.

(c) Since $\ker(A) = 0$ then the columns of A are linearly independent.
Since $\mathrm{im}(A) = \mathrm{span}\{\text{columns of } A\}$ and $\mathbb{Q}^t = \mathrm{im}(A)$ then $\mathbb{Q}^t = \mathrm{span}\{\text{columns of } A\}$.
So $\{\text{columns of } A\}$ is a basis of $\mathbb{Q}^t$.

(d) Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Q}^t$ have 1 in the ith entry and 0 elsewhere.
Then $\{e_1, \dots, e_t\}$ is a basis of $\mathbb{Q}^t$.
Let $\{a_1, \dots, a_s\}$ be the columns of A.
Then $\{a_1, \dots, a_s\}$ is a basis of $\mathbb{Q}^t$.
Since any two bases of $\mathbb{Q}^t$ have the same number of elements then $s = t$.

(e) To show: $P = Q$.
$P = P \cdot 1 = PAQ = 1 \cdot Q = Q$.
So $P = Q$. //