Example V6. Is $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ a $\mathbb{R}$-subspace of $\mathbb{R}^3$?

A $\mathbb{R}$-*subspace of* $\mathbb{R}^3$ is a subset $W \subseteq \mathbb{R}^3$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,

(b) $0 \in W$,

(c) If $w \in W$ then $-w \in W$,

(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof.

(a) Assume $w_1 = |a, b, c\rangle \in W$ and $w_2 = |x, y, z\rangle \in W$.

Then $a + b + c = 0$ and $x + y + z = 0$.

Then $w_1 + w_2 = |a + x, b + y, c + z\rangle$ and
$(a+x)+(b+y)+(c+z) = (a+b+c)+(x+y+z) = 0+0 = 0.$

So $w_1 + w_2 \in W$.

(b) $0 = |0, 0, 0\rangle$ satisfies $0 + 0 + 0 = 0$. So $0 \in W$.

(c) Assume $w = |x, y, z\rangle \in W$.

Then $x + y + z = 0$.

Then $-w = | -x, -y, -z\rangle$ and
$(-x) + (-y) + (-z) = -(x + y + z) = -0 = 0$.

So $-w \in W$.

(d) Assume $w = |x, y, z\rangle \in W$ and $c \in \mathbb{R}$.

Then $x + y + z = 0$.

Then $cw = |cx, cy, cz\rangle$ and
$cx + cy + cz = c(x + y + z) = c \cdot 0 = 0$.

So $cw \in W$.

So W is a subspace of $\mathbb{R}^3$. $\square$

Example V7. Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?

A *subspace of* $\mathbb{R}^2$ is a subset $L \subseteq \mathbb{R}^2$ such that

(a) If $w_1, w_2 \in L$ then $w_1 + w_2 \in L$,

(b) $0 \in L$,

(c) If $w \in L$ then $-w \in L$,

(d) If $w \in L$ and $c \in \mathbb{R}$ then $cw \in L$.

Since $0 = |0, 0\rangle$ and $0 \neq 2 \cdot 0 + 1$ then $0 \notin L$.
So L is not a subspace of $\mathbb{R}^2$.

Example V8. Is $W = \{a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}[x]_{\leqslant 2}$?

A *subspace of* $\mathbb{R}[x]_{\leqslant 2}$ is a subset $W \subseteq \mathbb{R}[x]_{\leqslant 2}$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,

(b) $0 \in W$,

(c) If $w \in W$ then $-w \in W$,

(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof.

(a) Assume $w_1 = a_1x + a_2x^2 \in W$ and $w_2 = b_1x + b_2x^2 \in W$.

Then $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

Then

$w_1 + w_2 = a_1x + a_2x^2 + b_1x + b_2x^2 = (a_1 + a_1)x + (b_1 + b_2)x^2$

and $a_1 + b_1 \in \mathbb{R}$ and $a_2 + b_2 \in \mathbb{R}$.

So $w_1 + w_2 \in W$.

(b) $0 = 0x + 0x^2$ satisfies $0 \in \mathbb{R}$ and $0 \in \mathbb{R}$. So $0 \in W$.

(c) Assume $w = a_1x + a_2x^2 \in W$.

Then $a_1, a_2 \in \mathbb{R}$.

Then $-w = -(a_1x + a_2x^2) = -a_1x + (-a_2)x^2$ and $-a_1 \in \mathbb{R}$ and $-a_2 \in \mathbb{R}$.

So $-w \in W$.

(d) Assume $w = a_1x + a_2x^2 \in W$ and $c \in \mathbb{R}$.

Then $a_1, a_2 \in \mathbb{R}$.

Then $cw = c(a_1x + a_2x^2) = (ca_1)x + (ca_2)x^2$ and $ca_1 \in \mathbb{R}$ and $ca_2 \in \mathbb{R}$.

So $cw \in W$.

So W is a subspace of $\mathbb{R}[x]_{\leqslant 2}$. $\square$

Example V9. Is the set of real 2×2 matrices whose trace is equal to 0 a subspace of $M_{2\times 2}(\mathbb{R})$?

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $W \subseteq M_{2\times 2}(\mathbb{R})$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,

(b) $0 \in W$,

(c) If $w \in W$ then $-w \in W$,

(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

Proof. The set of real 2×2 matrices whose trace is equal to 0 is

$$W = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ \middle|\ a_{11} + a_{22} = 0 \right\}.$$

(a) Assume $w_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $w_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in W$.

Then $a_{11} + a_{22} = 0$ and $b_{11} + b_{22} = 0$.

Then $w_1 + w_2 = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$ and

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0.$$

So $w_1 + w_2 \in W$.

(b) $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $(0, 0, 0)$ satisfies $0 + 0 = 0$. So $0 \in W$.

(c) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$.

Then $a_{11} + a_{22} = 0$.

Then $-w = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$ and

$(-a_{11}) + (-a_{22}) = -(a_{11} + a_{22}) = -0 = 0$.

So $-w \in W$.

(d) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $c \in \mathbb{R}$.

Then $a_{11} + a_{22} = 0$.

Then $cw = c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$ and

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = c \cdot 0 = 0.$$

So $cw \in W$.

So W is a subspace of $M_{2\times 2}(\mathbb{R})$. $\square$

Example V10. Is

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; ad - bc = 0 \right\} \text{ a subspace of } M_2(\mathbb{R})?.$$

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $S \subseteq M_{2\times 2}(\mathbb{R})$ such that

(a) If $w_1, w_2 \in S$ then $w_1 + w_2 \in S$,

(b) $0 \in S$,

(c) If $w \in S$ then $-w \in S$,

(d) If $w \in S$ and $c \in \mathbb{R}$ then $cw \in S$.

Let $w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $1 \cdot 0 - 0 \cdot 0 = 0 - 0 = 0$ then $w_1 \in S$.

Let $w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $0 \cdot 1 - 0 \cdot 0 = 0 - 0 = 0$ then $w_2 \in S$.

Then

$$w_1 + w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad 1 \cdot 1 - 0 \cdot 0 = 1.$$

So $w_1 + w_2 \notin S$.

So S is not a subspace of $M_{2\times 2}(\mathbb{R})$.

Lecture 30: Review – Linear transformation examples

Example LT3. Is the function $T\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \quad \text{a linear transformation?}$$

A *linear transformation from* $M_2(\mathbb{R})$ *to* $\mathbb{R}$ is a function $f\colon M_2(\mathbb{R}) \to \mathbb{R}$ such that

(a) If $v_1, v_2 \in M_2(\mathbb{R})$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,

(b) If $c \in \mathbb{R}$ and $v \in M_2(\mathbb{R})$ then $f(cv) = cf(v)$.

Since

$$1 = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is not equal to

$$0 = 0 + 0 = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then condition (a) does not hold and T is not a linear transformation.

Example LT4. Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_2 - 2x_3, 3x_1 + x_3) \qquad \text{a linear transformation?}$$

A *linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ is a function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

(a) If $u, v \in \mathbb{R}^3$ then $f(u+v) = f(u) + f(v)$,
(b) If $c \in \mathbb{R}$ and $v \in \mathbb{R}^3$ then $f(cv) = cf(v)$.

(a) Assume $u, v \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$ and $v = |v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}
T(|u_1, u_2, u_3\rangle + |v_1, v_2, v_3\rangle &= T(|u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle) \\
&= |(u_2 + v_2 - 2(u_3 + v_3), 3(u_1 + v_1) + (u_3 + v_3)\rangle \\
&= |u_2 - 2u_3 + v_2 - 2v_3, 3u_1 + u_3 + 3v_1 + v_3\rangle \\
&= |u_2 - 2u_3, 3u_1 + u_3\rangle + |v_2 - 2v_3, 3v_1 + v_3\rangle \\
&= T(|u_1, u_2, u_3\rangle) + T(|v_1, v_2, v_3\rangle)
\end{aligned}$$

(b) Assume $c \in \mathbb{R}$ and $u \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$. Then

$$\begin{aligned}
T(c \cdot |u_1, u_2, u_3\rangle) = T(|cu_1, cu_2, cu_3\rangle) &= |cu_2 - 2cu_3, 3cu_1 + cu_3\rangle \\
&= c|u_2 - 2u_3, 3u_1 + u_3\rangle = cT(|u_1, u_2, u_3\rangle).
\end{aligned}$$

So T is a linear transformation.

Example V12. In $\mathbb{R}^3$, is $|1,2,3\rangle \in \mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$?

By definition $\mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$

$$= \{c_1|1,-1,2\rangle + c_2|-1,1,2\rangle \mid c_1, c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$|1,2,3\rangle = c_1|1,-1,2\rangle + c_2|-1,1,2\rangle.$$

So we need to show that the system $\begin{matrix} c_1 - c_2 = 1, \\ -c_1 + c_2 = 2, \\ 2c_1 + 2c_2 = 3, \end{matrix}$ has a solution.

In matrix form the equations are $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 2 & 2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

Already this gives an equation $0c_1 + 0c_2 = 3$, which has no solution.

So $|1,2,3\rangle \notin \mathbb{R}\text{-span}\{|1,-1,2\rangle$ and $|-1,1,2\rangle\}$.

So $|1,2,3\rangle$ is not a linear combination of $|1,-1,2\rangle$ and $|-1,1,2\rangle$.

So $|1,2,3\rangle \notin \mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$. □

Example V13. In $\mathbb{R}[x]_{\leqslant 2}$, is $1-2x-x^2 \in \mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$?

By definition $\mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$

$$= \{c_1(1+x+x^2)+c_2(3+x^2) \mid c_1 c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1+x+x^2)+c_2(3+x^2) = 1-2x-x^2.$$

So we need to show that the system $\begin{aligned} c_1 + 3c_2 &= 1, \\ c_1 + 0c_2 &= -2, \\ c_1 + c_2 &= -1, \end{aligned}$ has a solution.

In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & -1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 3\\1 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}0 & 1 & 0\\1 & -1 & 0\\0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 1\\0 & 2\\0 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & -2\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 1\\0 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-1\\-1\\0\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

So $c_1 = -2$ and $c_2 = 1$ is a solution.

So $-2(1 + x + x^2) + (3 + x^2) = 1 - 2x - x^2$.

So $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$.

So $1 - 2x - x^2$ is a linear combination of $1 + x + x^2$ and $3 + x^2$. $\square$

Example V14. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1),(2,2,2),(3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

In this case

$$\begin{aligned}
\mathbb{R}\text{-span}(S) &= \{c_1\,|1,1,1\rangle + c_2\,|2,2,2\rangle + c_3\,|3,3,3\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{c_1\,|1,1,1\rangle + 2c_2\,|1,1,1\rangle + 3c_3\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{(c_1 + 2c_2 + 3c_3)\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{t\,|1,1,1\rangle \mid t \in \mathbb{R}\} \\
&= \{|t,t,t\rangle \mid t \in \mathbb{R}\}.
\end{aligned}$$

□

Example V15. Let S be the subset of $\mathbb{R}^2$ given by

$$S = \{|1,-1\rangle, |2,4\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^2.$$

Proof. By definition $\mathbb{R}\text{-span}(S) = \{c_1|1,-1\rangle + c_2|2,4\rangle \mid c_1, c_2 \in \mathbb{R}\}$.

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$
(b) $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar mutliplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$.

(b) To show: $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication, we can show $\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$c_1|1,-1\rangle + c_2|2,4\rangle = |1,0\rangle \quad \text{and} \quad d_1|1,-1\rangle + d_2|2,4\rangle = |0,1\rangle.$$

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\tfrac{2}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |1,0\rangle, \text{ and}$$

$$-\tfrac{1}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |0,1\rangle.$$

So $|1,0\rangle \in \mathbb{R}\text{-span}(S)$ and $|0,1\rangle \in \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$. $\square$

Example V16. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{|1,2,0\rangle, |1,5,3\rangle, |0,1,1\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^3.$$

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\}.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$.

(b) To show: $\mathbb{R}^3 \subseteq \mathrm{span}(S)$.

To show: $\mathbb{R}\text{-span}\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathrm{span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,

To show: $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle &= |1,0,0\rangle, \\ d_1|1,2,0\rangle + d_2|1,5,3\rangle + d_3|0,1,1\rangle &= |0,1,0\rangle, \\ r_1|1,2,0\rangle + r_2|1,5,3\rangle + r_3|0,1,1\rangle &= |0,0,1\rangle, \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\text{span}(S) \neq \mathbb{R}^2$. □

Example V17. Let S be the subset of $\mathbb{R}[x]_{\leqslant 2}$ given by

$$S = \{1 + x + x^2, x^2\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}[x]_{\leqslant 2}.$$

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1(1 + x + x^2) + c_2 x^2 \mid c_1, c_2 \in \mathbb{R}\}.$$

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{R}[x]_{\leqslant 2}$
(b) $\mathbb{R}[x]_{\leqslant 2} \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}[x]_{\leqslant 2}$ and $\mathbb{R}[x]_{\leqslant 2}$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\leqslant 2}$.

(b) To show: $\mathbb{R}[x]_{\leqslant 2} \subseteq \mathbb{R}\text{-span}(S)$.

To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,

To show: $\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2x^2 = 1, \quad d_1(1 + x + x^2) + d_2x^2 = x,$$

and

$$r_1(1 + x + x^2) + r_2x^2 = x^2.$$

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the top row on the left hand side is all 0 and the top row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}\text{-span}\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}[x]_{\leqslant 2} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}\text{-span}(S) \neq \mathbb{R}[x]_{\leqslant 2}$. □

Lecture 32: Review – Linear independence examples

Example V18a Let S be the subset of $\mathbb{C}^3$ given by

$$S = \{|2i, -1, 1\rangle, |-6, -3i, 3i\rangle\}. \qquad \text{Is } S\ \mathbb{C}\text{-linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$ then $c_1 = 0,\ c_2 = 0$.

Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$.

Then

$$\begin{aligned} 2ic_1 - 6c_2 &= 0, \\ -c_1 - 3ic_2 &= 0, \\ c_1 + 3ic_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2i & -6 \\ -1 & -3i \\ 1 & 3i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3i \\ 1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0,\ c_2 = 0$ is not the only solution.

So S is not linearly independent.

Example V18b. Let B be the subset of $\mathbb{R}^3$ given by

$$B = \{|2i, -1, 1\rangle, |4, 0, 2\rangle\}. \qquad \text{Is } B \text{ linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$ then $c_1 = 0,\ c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$
Then

$$\begin{aligned} 2ic_1 + 4c_2 &= 0, \\ -c_1 + 0c_2 &= 0, \\ c_1 + 2c_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2i & 4 \\ -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution $c_1 = 0,\ c_2 = 0$.
So S is linearly independent.

Example V19. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2,0,0),(6,1,7),(2,-1,2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show:
If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$
then $c_1 = 0,\ c_2 = 0,\ c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} 2c_1 + 6c_2 + 2c_3 &= 0,\\ c_2 - c_3 &= 0,\\ 7c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or equivalently} \quad \begin{pmatrix} 2 & 6 & 2\\ 0 & 1 & -1\\ 0 & 7 & 2 \end{pmatrix} \begin{pmatrix} c_1\\ c_2\\ c_3 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution:
$c_1 = 0,\ c_2 = 0,\ c_3 = 0$.
So S is linearly independent.

Example V20&26. Let S be the subset of $\mathbb{R}[x]_{\leqslant 2}$ given by

$S = \{1 + 2x + 5x^2, 1 + x + x^2, 1 + 2x + 3x^2\}$. Is S a basis of $\mathbb{R}[x]_{\leqslant 2}$?

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$
then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Assume $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$.
Then

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ 2c_1 + c_2 + 2c_3 &= 0, \\ 5c_1 + c_2 + 3c_2 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution:
$c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent.

Since $\dim(\mathbb{R}[x]_{\leqslant 2}) = 3$ and S contains 3 linearly independent elements then B is a basis for $\mathbb{R}[x]_{\leqslant 2}$.