Lecture 30: Review – Linear transformation examples

Example LT3. Is the function $T\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \quad \text{a linear transformation?}$$

A *linear transformation from $M_2(\mathbb{R})$ to $\mathbb{R}$* is a function $f\colon M_2(\mathbb{R}) \to \mathbb{R}$ such that

(a) If $v_1, v_2 \in M_2(\mathbb{R})$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,

(b) If $c \in \mathbb{R}$ and $v \in M_2(\mathbb{R})$ then $f(cv) = cf(v)$.

Since

$$1 = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is not equal to

$$0 = 0 + 0 = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then condition (a) does not hold and T is not a linear transformation.

Example LT4. Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_2 - 2x_3, 3x_1 + x_3) \qquad \text{a linear transformation?}$$

A *linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ is a function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

(a) If $u, v \in \mathbb{R}^3$ then $f(u+v) = f(u) + f(v)$,
(b) If $c \in \mathbb{R}$ and $v \in \mathbb{R}^3$ then $f(cv) = cf(v)$.

(a) Assume $u, v \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$ and $v = |v_1, v_2, v_3\rangle$. Then

$$\begin{aligned} T(|u_1, u_2, u_3\rangle + |v_1, v_2, v_3\rangle &= T(|u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle) \\ &= |(u_2 + v_2 - 2(u_3 + v_3), 3(u_1 + v_1) + (u_3 + v_3))\rangle \\ &= |u_2 - 2u_3 + v_2 - 2v_3, 3u_1 + u_3 + 3v_1 + v_3\rangle \\ &= |u_2 - 2u_3, 3u_1 + u_3\rangle + |v_2 - 2v_3, 3v_1 + v_3\rangle \\ &= T(|u_1, u_2, u_3\rangle) + T(|v_1, v_2, v_3\rangle) \end{aligned}$$

(b) Assume $c \in \mathbb{R}$ and $u \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$. Then

$$\begin{aligned} T(c \cdot |u_1, u_2, u_3\rangle) = T(|cu_1, cu_2, cu_3\rangle) &= |cu_2 - 2cu_3, 3cu_1 + cu_3\rangle \\ &= c|u_2 - 2u_3, 3u_1 + u_3\rangle = cT(|u_1, u_2, u_3\rangle). \end{aligned}$$

So T is a linear transformation.

Lecture 31: Review – Span examples

Example V12. In $\mathbb{R}^3$, is $|1,2,3\rangle \in \mathbb{R}$-span$\{|1,-1,2\rangle, |-1,1,2\rangle\}$?

By definition $\mathbb{R}$-span$\{|1,-1,2\rangle, |-1,1,2\rangle\}$

$$= \{c_1|1,-1,2\rangle + c_2|-1,1,2\rangle \mid c_1, c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$|1,2,3\rangle = c_1|1,-1,2\rangle + c_2|-1,1,2\rangle.$$

So we need to show that the system $\begin{aligned} c_1 - c_2 &= 1, \\ -c_1 + c_2 &= 2, \\ 2c_1 + 2c_2 &= 3, \end{aligned}$ has a solution.

In matrix form the equations are $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 2 & 2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

Already this gives an equation $0c_1 + 0c_2 = 3$, which has no solution.

So $|1,2,3\rangle \notin \mathbb{R}$-span$\{|1,-1,2\rangle$ and $|-1,1,2\rangle\}$.

So $|1,2,3\rangle$ is not a linear combination of $|1,-1,2\rangle$ and $|-1,1,2\rangle$.

So $|1,2,3\rangle \notin \mathbb{R}$-span$\{|1,-1,2\rangle, |-1,1,2\rangle\}$. $\square$

Example V13. In $\mathbb{R}[x]_{\leqslant 2}$, is $1-2x-x^2 \in \mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$?

By definition $\mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$

$$= \{c_1(1+x+x^2) + c_2(3+x^2) \mid c_1 c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1+x+x^2) + c_2(3+x^2) = 1-2x-x^2.$$

So we need to show that the system $\begin{aligned} c_1 + 3c_2 &= 1, \\ c_1 + 0c_2 &= -2, \\ c_1 + c_2 &= -1, \end{aligned}$ has a solution.

In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

So $c_1 = -2$ and $c_2 = 1$ is a solution.
So $-2(1 + x + x^2) + (3 + x^2) = 1 - 2x - x^2$.
So $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$.
So $1 - 2x - x^2$ is a linear combination of $1 + x + x^2$ and $3 + x^2$. $\square$

Example V14. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1),(2,2,2),(3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

In this case

$$\begin{aligned}
\mathbb{R}\text{-span}(S) &= \{c_1\,|1,1,1\rangle + c_2\,|2,2,2\rangle + c_3\,|3,3,3\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{c_1\,|1,1,1\rangle + 2c_2\,|1,1,1\rangle + 3c_3\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{(c_1 + 2c_2 + 3c_3)\,|1,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\} \\
&= \{t\,|1,1,1\rangle \mid t \in \mathbb{R}\} \\
&= \{|t,t,t\rangle \mid t \in \mathbb{R}\}.
\end{aligned}$$

$\square$

Example V15. Let S be the subset of $\mathbb{R}^2$ given by

$$S = \{|1, -1\rangle, |2, 4\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^2.$$

Proof. By definition $\mathbb{R}\text{-span}(S) = \{c_1|1, -1\rangle + c_2|2, 4\rangle \mid c_1, c_2 \in \mathbb{R}\}$.

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$
(b) $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar mutliplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$.

(b) To show: $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{|1, 0\rangle, |0, 1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication, we can show $\{|1, 0\rangle, |0, 1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$c_1|1, -1\rangle + c_2|2, 4\rangle = |1, 0\rangle \quad \text{and} \quad d_1|1, -1\rangle + d_2|2, 4\rangle = |0, 1\rangle.$$

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\tfrac{2}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |1,0\rangle, \text{ and}$$

$$-\tfrac{1}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |0,1\rangle.$$

So $|1,0\rangle \in \mathbb{R}\text{-span}(S)$ and $|0,1\rangle \in \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$. $\square$

Example V16. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{|1,2,0\rangle, |1,5,3\rangle, |0,1,1\rangle\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}^3.$$

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\}.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$.

(b) To show: $\mathbb{R}^3 \subseteq \mathrm{span}(S)$.

To show: $\mathbb{R}\text{-span}\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathrm{span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,

To show: $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle &= |1,0,0\rangle, \\ d_1|1,2,0\rangle + d_2|1,5,3\rangle + d_3|0,1,1\rangle &= |0,1,0\rangle, \\ r_1|1,2,0\rangle + r_2|1,5,3\rangle + r_3|0,1,1\rangle &= |0,0,1\rangle, \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathrm{span}(S) \neq \mathbb{R}^2$. $\square$

Example V17. Let S be the subset of $\mathbb{R}[x]_{\leqslant 2}$ given by

$$S = \{1 + x + x^2, x^2\}. \qquad \text{Show that } \operatorname{span}(S) = \mathbb{R}[x]_{\leqslant 2}.$$

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1(1 + x + x^2) + c_2x^2 \mid c_1, c_2 \in \mathbb{R}\}.$$

To show: (a) $\operatorname{span}(S) \subseteq \mathbb{R}[x]_{\leqslant 2}$
(b) $\mathbb{R}[x]_{\leqslant 2} \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}[x]_{\leqslant 2}$ and $\mathbb{R}[x]_{\leqslant 2}$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\leqslant 2}$.

(b) To show: $\mathbb{R}[x]_{\leqslant 2} \subseteq \mathbb{R}\text{-span}(S)$.

To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,

To show: $\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2x^2 = 1, \quad d_1(1 + x + x^2) + d_2x^2 = x,$$

and

$$r_1(1 + x + x^2) + r_2x^2 = x^2.$$

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the top row on the left hand side is all 0 and the top row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}\text{-span}\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}[x]_{\leqslant 2} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathbb{R}\text{-span}(S) \neq \mathbb{R}[x]_{\leqslant 2}$. $\square$