Lecture 32: Review – Linear independence examples

Example V18a Let S be the subset of $\mathbb{C}^3$ given by

$$S = \{|2i, -1, 1\rangle, |-6, -3i, 3i\rangle\}. \qquad \text{Is } S \ \mathbb{C}\text{-linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$ then $c_1 = 0$, $c_2 = 0$.

Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$.

Then

$$\begin{aligned} 2ic_1 - 6c_2 &= 0, \\ -c_1 - 3ic_2 &= 0, \\ c_1 + 3ic_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2i & -6 \\ -1 & -3i \\ 1 & 3i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3i \\ 1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0$, $c_2 = 0$ is not the only solution.

So S is not linearly independent.

Example V18b. Let B be the subset of $\mathbb{R}^3$ given by

$$B = \{|2i, -1, 1\rangle, |4, 0, 2\rangle\}. \qquad \text{Is } B \text{ linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$ then $c_1 = 0$, $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$
Then

$$\begin{aligned} 2ic_1 + 4c_2 &= 0, \\ -c_1 + 0c_2 &= 0, \\ c_1 + 2c_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2i & 4 \\ -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution $c_1 = 0$, $c_2 = 0$.
So S is linearly independent.

Example V19. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2,0,0), (6,1,7), (2,-1,2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show:
If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$
then $c_1 = 0,\ c_2 = 0,\ c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} 2c_1 + 6c_2 + 2c_3 &= 0, \\ c_2 - c_3 &= 0, \\ 7c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or equivalently} \quad \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & -1 \\ 0 & 7 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution:
$c_1 = 0,\ c_2 = 0,\ c_3 = 0$.
So S is linearly independent.

Example V20&26. Let S be the subset of $\mathbb{R}[x]_{\leqslant 2}$ given by

$S = \{1 + 2x + 5x^2, 1 + x + x^2, 1 + 2x + 3x^2\}$. Is S a basis of $\mathbb{R}[x]_{\leqslant 2}$?

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$
then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Assume $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$.
Then

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ 2c_1 + c_2 + 2c_3 &= 0, \\ 5c_1 + c_2 + 3c_2 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution:
$c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent.

Since $\dim(\mathbb{R}[x]_{\leqslant 2}) = 3$ and S contains 3 linearly independent elements then B is a basis for $\mathbb{R}[x]_{\leqslant 2}$.

Example V21. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and
$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$
Then

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 0, \\ 3c_1 + c_2 + 10c_3 &= 0, \\ c_1 + c_2 + 4c_3 &= 0, \\ c_1 - c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ is not the only solution.
So S is not linearly independent.

Here is a check that $c_1 = -3$, $c_2 = 1$, $c_3 = -1$ is a solution:

$$\begin{aligned} -3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} &= -3 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -9 \\ -3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Lecture 33: Review – Basis examples

Example V23. Is $S = \{(1,-1),(2,4)\}$ a basis of $\mathbb{R}^2$?
Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \qquad \text{Then} \qquad A^{-1} = \tfrac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{gives} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So S is linearly independent.

If $|a,b\rangle \in \mathbb{R}^2$ then $|a,b\rangle = c_1|1,-1\rangle + c_2|2,4\rangle$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3}a - \frac{1}{3}b \\ \frac{1}{6}a + \frac{1}{6}b \end{pmatrix}.$$

So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$. Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$. So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$.

So S is a basis of $\mathbb{R}^2$.

Example V24. Is $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ a basis of

$$\{A \in M_2(\mathbb{R}) \mid \operatorname{Tr}(A) = 0\}?$$

If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{aligned} c_1 &= 0, \\ c_2 &= 0, \\ c_3 &= 0. \end{aligned}$$

So S is linearly independent.

Then

$$
\begin{aligned}
&\{A \in M_2(\mathbb{R}) \mid \operatorname{Tr}(A) = 0\} \\
&= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}, a_{11} + a_{22} = 0 \right\} \\
&= \left\{ \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \\
&= \left\{ c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R} \right\} \\
&= \mathrm{span}(S).
\end{aligned}
$$

So S is a basis of $\{A \in M_2(\mathbb{R}) \mid \operatorname{Tr}(A) = 0\}$.
Example V25. Is

$$
S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \qquad \text{a basis of } M_2(\mathbb{R})?
$$

Since $E_{11}, E_{12}, E_{21}, E_{22}$ is a basis of $M_2(\mathbb{R})$ then $\dim(M_2(\mathbb{R})) = 4$.
Since S contains only 3 elements then S is not a basis of $M_2(\mathbb{R})$.