Lecture 36: Tutorial: Inverses of an arbitrary 2×2 matrix

Example A1 Let $a, b, c, d \in \mathbb{Q}$ and find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Start with $AA^{-1} = 1$, which is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 1: $c \neq 0$. Left multiply by $s_1(\frac{a}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -\frac{a}{c} \end{pmatrix}, \quad \text{to get} \begin{pmatrix} c & d \\ 0 & b - \frac{a}{c}d \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{a}{c} \end{pmatrix}.$$

Case 1a: $c \neq 0$ and $ad - bc \neq 0$. Left multiply by $h_1(c)^{-1} h_2(\frac{bc-ad}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \frac{c}{bc-ad} \end{pmatrix}, \quad \text{to get} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix}.$$

Left multiply by $x_{12}(\frac{d}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix}, \quad \text{to get} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -\frac{d}{bc-ad} & \frac{1}{c} + \frac{ad}{c(bc-ad)} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix}.$$

So

$$A^{-1} = \begin{pmatrix} -\frac{d}{bc-ad} & \frac{bc-ad+ad}{c(bc-ad)} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Case 1b: $c \neq 0$ and $ad - bc = 0$. Then

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{a}{c} \end{pmatrix}$$

and there does not exist any matrix A^{-1} that makes this equation true.

Case 2: $c = 0$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 2a: $c = 0$ and $d \neq 0$ and $a \neq 0$.
Left multiply by $h_2(d)^{-1} h_1(a)^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}.$$

Left mulltiply by $x_{12}(\frac{b}{a})^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} \frac{d}{ad} & \frac{-b}{ad} \\ 0 & \frac{a}{ad} \end{pmatrix}.$$

Recalling that $c = 0$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem (Inverse of a 2×2 matrix)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{Q})$. *Then*

1. *If* $ad - bc \neq 0$ *then* $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
2. *If* $ad - bc = 0$ *then* A^{-1} *does not exist.*

Example M5. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{(2\cdot 1 - 1\cdot(-1))}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{3} & 0 \\ 0 & \frac{3}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$